ISRAEL JOURNAL OF MATHEMATICS, Vol. 41, Nos. 1-2, 1982

GENERALIZED SMALL CANCELLATION THEORY AND APPLICATIONS I. THE WORD PROBLEM

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ABSTRACT
In this paper we develop a generalization of the small cancellation theory. The usual small cancellation hypotheses are replaced by some condition that, roughly speaking, says that if a common part of two relations is a big piece of one relation then it must be a very small piece of another. In particular, we show that a finitely presented generalized small cancellation group has a solvable word problem. The machinery developed in the paper is to be used in the following papers of this series for constructing some group-theoretic examples.

Introduction

Various problems in group theory are related to construction of groups by generators and relations. Although most algorithmic problems concerning presentations of groups (in particular, even the problem of being trivial) have, in general, a negative solution, it has been discovered that, in certain cases, important information about a group can be derived from the combinatorial properties of its presentation by generators and defining relations.

Max Dehn solved the word and conjugacy problems for the fundamental groups of compact Riemann surfaces of genus > 1. These groups are defined by a single relator r with the property that, if s is a cyclic permutation of r or r^{-1}, with $s \neq r^{-1}$, there is very little cancellation when the product rs is formed. Dehn's results were later generalized by several authors to a wider class of groups, possessing presentations in which the defining relations have a similar small cancellation property (for more details and bibliography, see [1]).

An essential feature of small-cancellation groups is that, if a freely reduced non-trivial word w is equal to 1, then w contains a large part of some cyclic permutation of a defining relation (or its inverse). This yields a criterion for $w \neq 1$, which is used to prove some embedding theorems by small cancellation

Received March 20, 1980

methods (see [1], p. 282). Moreover, this criterion suggests that small cancellation may prove helpful in showing that a given group is non-trivial or even infinite. However, in trying to apply the small cancellation theory to certain group theoretic problems we meet difficulties, indicating that an essential generalization of the small cancellation hypotheses is needed. This is evident from the following example.

In order to construct a non-trivial finitely generated divisible group, it is natural to proceed as follows:

Let F be a finitely generated free group. Since the set $F \times \mathbf{N}$ is countable, we write its elements in a sequence

$$(g_1, n_1), (g_2, n_2), \cdots, (g_k, n_k), \cdots.$$

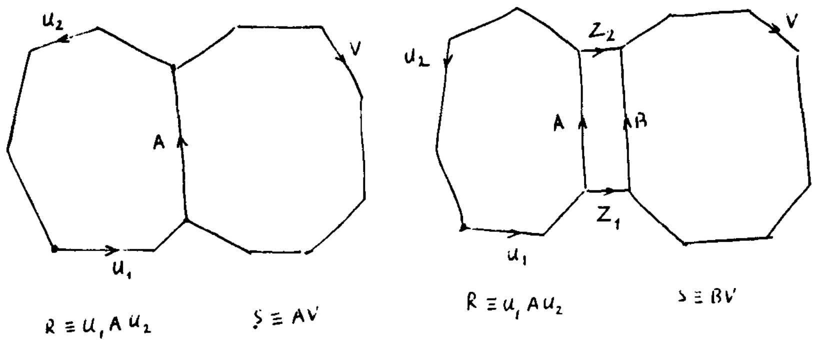
Choose $h_1, h_2, \cdots, h_k, \cdots$ elements of F and let $\mathcal{R}$ be the set of elements $\{h_k^{n_k} g_k^{-1} \mid k = 1, 2, \cdots\}$. For $N = \langle \mathcal{R} \rangle^F$, F/N is a finitely generated divisible group. The problem now reduces to verifying that, for a suitable choice of elements h_k, F/N is non-trivial. One is tempted to try to choose the elements h_k so that (after symmetrizing) the set $\mathcal{R} = \{h_k^{n_k} g_k^{-1} \mid k = 1, 2, \cdots\}$ satisfies suitable small cancellation conditions. Unfortunately, this seems to be impossible. Indeed, if $h_k^{n_k} g_k^{-1}$ is a cyclically reduced word, then the symmetrizing process adds $h_k^{n_k - 1} g_k^{-1} h_k$ to the relations and then $h_k^{n_k - 1}$ is a common initial segment of these two relations. If $h_k^{n_k} g_k^{-1}$ is reducible, a similar argument applies after this word has been reduced. Even worse, g_l ranges over all elements of F, and so, for any fixed $h_k^{n_k} g_k^{-1}$, some later $h_l^{n_l} g_l^{-1}$ will contain it as a segment.

Inevitably, we need either a different approach or a modification of the small cancellation hypotheses, in such a way that in certain cases relations having large common segments are admitted.

This is the objective of the first paper of this series, in which we introduce the following extension of the small cancellation hypotheses:

(1) We consider $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$ as a union of disjoint sets where, roughly speaking, the length of the words in $\mathcal{R}_n$ increases with n.

(2) We replace the notion of a piece (a common subword of two relations) by a new and more complicated notion which relates subwords of relations. Graphically, the comparison between the old and the new notion is presented in Fig. 1 where words are denoted by lines. Here A is a subword of $R \in \mathcal{R}_k$, B is a subword of some relator $S \in \mathcal{R}_j$ or, possibly, of a power S^m of S ($m > 1$), Z_1 and Z_2 are words of a special type (they belong to the class of words $\mathcal{W}_h$ described in §1, where $h = \min(k, j) - 1$), $A^{-1} Z_1 B Z_2^{-1}$ belongs to the normal subgroup of F generated by $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_h$. We call A a (generalized) j-piece of R.



old notion new notion

Fig. 1.

(3) The condition that the generalized pieces be small, which is stated here in a metric form, can be formulated as follows:

S(λ, θ). Let λ and θ be two constants satisfying the inequalities

$$0<\lambda \leqq \frac{1}{21}, \qquad 0<\theta \leqq 1-\frac{6\lambda + 13\lambda^2}{1-13\lambda}.$$

For a (generalized) j-piece A of a relator $R \in \mathcal{R}_k$ we require that, if $j > k$, then $|A| < \lambda^{k-j+1}|R|$, and, if $j > k$, then $|A| < \theta|R|$, where $|W|$ denotes the length of the word W.

As a matter of fact, in the paper we shall use the following, closely related, non-metric condition.

(S) Let $R \in \mathcal{R}_k$ be decomposed as a product of generalized pieces of various types:

$$R \equiv A_1 A_2 \cdots A_p.$$

For $j = 1, 2, \cdots$, let d_j be the number of (generalized) j-pieces A_l appearing in this factorization. Then the numbers $d_1, d_2, \cdots$ are subject to the following limitations:

(α) We cannot have

$$d_1 \leqq 8\cdot 13^{k-1}, \quad d_2 \leqq 8\cdot 13^{k-2}, \cdots, d_k \leqq 8, \quad d_j = 0 \qquad \text{for } j > k.$$

(β) We cannot have, for some $h > k$,

$$d_1 \leqq 7\cdot 13^{k-1}, \quad d_2 \leqq 7\cdot 13^{k-2}, \cdots, d_{k-1} \leqq 7\cdot 13,$$

$$d_k \leqq 6, \quad d_h = 1 \quad \text{and} \quad d_j = 0 \qquad \text{for } j > k,\, j \neq h.$$

It can be shown that $S(\lambda, \theta)$ implies (S). Indeed, if $S(\lambda, \theta)$ holds, then

$$|R| = \sum_{e=1}^{p} |A_e| < \left(\sum_{j=1}^{k} d_j \lambda^{k-j+1} + \theta \cdot \sum_{j>k} d_j\right) |R|.$$

If $d_j \leqq 8\cdot 13^{k-j}$ for $j \leqq k$ and $d_j = 0$ for $j > k$ then, for $\lambda \leqq 1/21$,

$$|R| < \left(\sum_{j=1}^{k} 8\cdot 13^{k-j} \lambda^{k-j+1}\right) |R| < \frac{8\lambda}{1-13\lambda} |R| \leqq |R|$$

which is a contradiction.

Now suppose that for some $h > k$, we have $d_j < 7\cdot 13^{k-j}$ for $j < k$, $d_k \leqq 6$, $d_h = 1$ and $d_j = 0$ for $j > k$, $j \neq h$; then

$$|R| < \left(\sum_{j=1}^{k-1} 7\cdot 13^{k-j} \lambda^{k-j+1} + 6\lambda + \theta\right) |R| < \left(\frac{6\lambda + 13\lambda^2}{1-13\lambda} + \theta\right) |R| \leqq |R|$$

which is also impossible. Thus, (S) holds.

Our main results can be stated in the metric form as follows (cf. Theorem 1, where the results are stated in the non-metric form):

Let $\mathcal{R}$ be a symmetrized subset of F, $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$, satisfying condition $S(\lambda, \theta)$. Let $N = \langle \mathcal{R} \rangle^F$. Then:

(1) Every (freely reduced) non-trivial word W in N contains a subword A which is related to a word B such that either B is a "large" subword of some relator R in $\mathcal{R}_i$ (i.e. $|B| > (1 - (4\lambda + 13\lambda^2)/(1 - 13\lambda))|R|$) or even $B \equiv R^m R'$ with $m \geqq 1$, $R \equiv R'R''$ in the following sense (see Fig. 2):

There are words Z_1, Z_2 of a special type (belonging to the class of words $\mathcal{W}_{i-1}$ described in §1 and satisfying condition (L)) such that $A^{-1}Z_1^{-1}BZ_2$ belongs to the normal subgroup of F generated by $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{i-1}$.

(2) Every (freely reduced) non-trivial word W in N contains a subword C which is also a subword of some relator $S \in \mathcal{R}_k$ such that $|C| > (1 - \theta - (4\lambda + 13\lambda^2)/(1 - 13\lambda))|S|$ (see Fig. 3).

(3) If $\mathcal{R}$ is finite then the quotient group F/N has a solvable group problem.

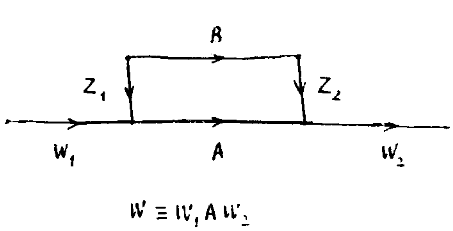


Fig. 2.

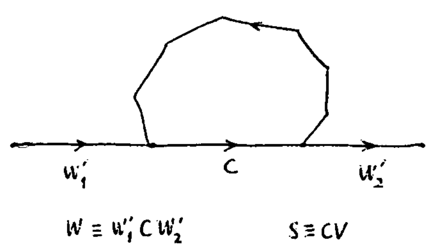


Fig. 3.

If $\mathcal{R}$ satisfies some (relatively mild) additional conditions, then one can deduce from part (1) the existence of an analog of Dehn's Algorithm in F/N (see Theorem 2).

Statement (2) implies the infinity of F/N in most cases. It is fundamental for the applications, which will be presented in the subsequent papers of this series.

Following the geometric approach of R. C. Lyndon, we consider van Kampen diagrams. We introduce a rank function on regions as follows: $\mathrm{rank}(\Phi)=i$ whenever the relator R written on the boundary of the region Φ belongs to $\mathcal{R}_i$. This makes it possible to translate our statements into statements about maps in the plane with a given rank of regions, subject to certain conditions of a combinatorial geometric nature.

§1. Statement of the main results; comments

1.1. Let F be a free group on a set X of generators. A *letter* is an element of the set Y of generators and inverses of generators. A *word* W is a finite string of letters, $W = y_1 \cdots y_n$. We denote the identity of F by 1. Each element of F has a unique presentation as a *reduced word* $W = y_1 \cdots y_n$ in which no two successive letters $y_j y_{j+1}$ form an inverse pair $x_i x_i^{-1}$ or $x_i^{-1} x_i$. The integer n is the *length* of W, which we denote by $|W|$. A reduced word W is said to be *cyclically reduced* if y_n is not the inverse of y_1. We use "$\equiv$" to denote graphical identity of words. The notation $U = V \pmod N$ means that the words U and V are equal modulo the normal subgroup N.

A subset $\mathcal{R}$ of F is said to be *symmetrized* if all elements of $\mathcal{R}$ are cyclically reduced and, for each R in $\mathcal{R}$, all cyclically reduced conjugates of both R and R^{-1} also belong to $\mathcal{R}$.

1.2. Let $(\mathcal{R}_i)_{i\geq 1}$ be a family of disjoint symmetrized subsets of F. We shall consider combinatorial conditions on this family which generalizes the small cancellation hypotheses.

These conditions depend on an auxiliary family of sets $(\mathcal{W}_i)_{i\geq 0}$.

Let N_i, $i = 1, 2, \cdots$, be the normal subgroup of F generated by $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_i$, let $N_0 = E$, the trivial subgroup, and let N be the normal subgroup of F generated by $\mathcal{R} = \bigcup_{i\geq 1} \mathcal{R}_i$.

We are going to generalize the notion of a piece of a relator. Our starting point is the following definition of a piece in the ordinary small cancellation theory.

A subword A of a relator $R \equiv U_1 A U_2$ is said to be a piece if there is a relator S, with a factorization $S \equiv AV$, such that $S^{-1} A U_2 U_1$ is not freely equal to 1 or to a conjugate of a relator (see [1], p. 240 and p. 271).

DEFINITION 1. Given an integer $j \geqq 1$, a word A is said to be a (*generalized*) *j-piece of a relator $R \in \mathcal{R}_k$ (relative to $(\mathcal{R}_i)_{i\geqq 1}$ and $(\mathcal{W}_i)_{i\geqq 0}$)* if $R \equiv U_1 A U_2$ and there exist a relator $S \in \mathcal{R}_j$ and two words $Z_1, Z_2 \in \mathcal{W}_h$, where $h = \min(k, j) - 1$, such that (see Fig. 1):

(1) For some $m \geqq 1$, there is a factorization $S^m \equiv BV$.

(2) $A = Z_1 B Z_2^{-1} \pmod{N_h}$.

(3) If $k = j$ then (α) $Z_1 S^{-1} Z_1^{-1} A U_2 U_1 \notin N_h$; (β) $Z_1 S^{-1} Z_1^{-1} A U_2 U_1$ is not conjugate modulo N_h to a relator $T \in \mathcal{R}_k$.

Let $\mathcal{P}(R; j)$ denote the set of all j-pieces of a relator R.

We shall use factorizations of subwords of relators into products of generalized pieces of various types. In this connection we introduce the following notation.

Let $c = (c_1, c_2, \cdots)$ be a sequence of numbers. For a relator R, $\mathcal{I}(R; c)$ will denote the set of all subwords D of R^n, i.e. $R^n \equiv P_1 D P_2$, which have a factorization $D \equiv D_1 D_2 \cdots D_k$ such that each D_l is an $f(l)$-piece of R, $1 \leqq l \leqq k$, and

$$\operatorname{card}\{l \mid f(l) = j\} \leqq c_j \qquad (j \geqq 1)$$

(i.e., the number of j-pieces in this factorization does not exceed c_j). $\mathcal{H}(R; c)$ will denote the set of all subwords Q of R^n such that every subword of Q belongs to $\mathcal{I}(R; c)$ $(n \geqq 1)$.

1.3. Introducing sequences $e_j = (0, 0, \cdots, 0, 1, 0, \cdots)$, where 1 is in the j-th place, we can write $c = \Sigma_{j \geqq 1} c_j e_j$.

Our generalized small cancellation hypotheses consist of two conditions (S) and (L), which we now state.

Condition (S). For any $i \geqq 1$ and $R \in \mathcal{R}_i$,

(α) $R \notin \mathcal{I}(R; \Sigma_{j=1}^{i} 8 \cdot 13^{i-j} e_j)$;

(β) for any $k > i$, $R \notin \mathcal{I}(R; \Sigma_{j=1}^{i-1} 7 \cdot 13^{i-j} e_j + 6 e_i + e_k)$.

REMARK. This condition means that, if R has a factorization $R \equiv D_1 D_2 \cdots D_h$ into a product of generalized pieces D_l, $1 \leqq l \leqq h$, then (α) asserts that it cannot happen that none of the D_l's is a j-piece for $j > i$ and that, at the same time, there are at most $8 \cdot 13^{i-j}$ j-pieces for $j = 1, 2, \cdots, i$; or, stated positively, either some D_l is a j-piece with $j > i$, or for some j, $1 \leqq j \leqq i$, there are more than $8 \cdot 13^{i-j}$ j-pieces in the factorization. Similarly, (β) asserts that it cannot happen that only one D_l is a k-piece with $k > i$, all other factors are j-pieces with $j \leqq i$, the number of i-pieces does not exceed 6, and, for $j < i$, the number of j-pieces does not exceed $7 \cdot 13^{i-j}$.

Roughly speaking, condition (S) states that, for any relator $R \in \mathcal{R}_i$, the j-pieces of R with $j \leqq i$ are "relatively small" subwords, while the j-pieces of R with $j > i$ are "strictly less" than R (and cannot be completed to R even by adding "relatively many" generalized pieces of types $\leqq i$).

Condition (L). (α) $1 \in \mathcal{W}_i$ for all $i \geqq 0$;

(β) if $U_1, U_2 \in \mathcal{W}_{i-1}$ and $V \in \mathcal{H}(R; \sum_{j=1}^{i-1} 2\cdot 13^{i-j} e_j + e_i)$ for some $R \in \mathcal{R}_i$, then $U_1 V U_2 \in \mathcal{W}_i$, $i = 1, 2, \cdots$.

Remark. Notice that, according to Definition 1, the larger the sets $\mathcal{W}_i$, the more possibilities we have for generalized pieces, hence the larger are the sets $\mathcal{P}(R; j)$, $\mathcal{I}(R; c)$, $\mathcal{H}(R; c)$ and the more restrictive is condition (S).

1.4. Our main result is the following

Theorem 1. *Let $(\mathcal{R}_i)_{i \geq 1}$ be a family of disjoint symmetrized subsets of the free group F and let $(\mathcal{W}_i)_{i \geq 0}$ be a family of subsets of F. Let $N(N_i)$ denote the normal subgroup of F generated by $\mathcal{R} = \bigcup_{i \geq 1} \mathcal{R}_i$ (respectively, by $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_i$).*

If the families of sets $(\mathcal{R}_i)_{i \geq 1}$ and $(\mathcal{W}_i)_{i \geq 0}$ satisfy conditions (S) *and* (L) *then:*

(i) *Every freely reduced non-trivial word W in N contains a subword A (i.e. $W \equiv W_1 A W_2$) for which there exist a word B, an integer i, two words $Z_1, Z_2 \in \mathcal{W}_{i-1}$ and a relator $R \in \mathcal{R}_i$ such that*

$$A^{-1} Z_1^{-1} B Z_2 \in N_{i-1}$$

and either there exists a factorization $R \equiv BU$ with

$$U \in \mathcal{H}\left(R; \sum_{j=1}^{i-1} 5\cdot 13^{i-j} e_j + 4e_i\right)$$

or $B \equiv R^m R'$, with $m \geqq 1$ and $R \equiv R'R''$ (see Fig. 2).

(ii) *Every freely reduced non-trivial word W in N contains a subword C (i.e. $W \equiv W_1' C W_2'$) for which there exist an integer k and a relator $S \in \mathcal{R}_k$ with a factorization $S \equiv CV$ such that either*

$$V \in \mathcal{H}\left(S; \sum_{j=1}^{k} 4\cdot 13^{k-j} e_j\right)$$

or, for some $h > k$,

$$V \in \mathcal{H}\left(S; \sum_{j=1}^{k-1} 3\cdot 13^{k-j} e_j + 2e_k + e_h\right) \qquad \text{(see Fig. 3).}$$

(iii) *If $\bigcup_{i \geq 1} \mathcal{R}_i$ is finite then $G = F/N$ has a solvable word problem.*

By imposing the metric condition $S(\lambda, \theta)$, we obtain information about the relative lengths of B and C:

COROLLARY 1. *If* $(\mathcal{R}_i)_{i\geqq 1}$ *and* $(\mathcal{W}_i)_{i\geqq 0}$ *satisfy* $S(\lambda, \theta)$ *and* (L) *then in the notation of part* (i) *of Theorem* 1 *we also have*

$$|B| > \left(1 - \frac{4\lambda + 13\lambda^2}{1 - 13\lambda}\right)|R|$$

and in the notation of part (ii) *of Theorem* 1 *we have*

$$|C| > \left(1 - \theta - \frac{4\lambda + 13\lambda^2}{1 - 13\lambda}\right)|S|.$$

PROOF. As shown in the introduction, condition $S(\lambda, \theta)$ implies condition (S). By part (i) of Theorem 1,

$$U \in \mathcal{H}\left(R; \sum_{j=1}^{i-1} 5\cdot 13^{i-j} e_j + 4e_i\right).$$

Hence, by $S(\lambda, \theta)$,

$$|U| < \left(\sum_{j=1}^{i-1} 5\cdot 13^{i-j}\lambda^{i-j+1} + 4\lambda\right)|R| < \frac{4\lambda + 13\lambda^2}{1 - 13\lambda}|R|,$$

and then

$$|B| = |R| - |U| > \left(1 - \frac{4\lambda + 13\lambda^2}{1 - 13\lambda}\right)|R|.$$

Similarly, from part (ii) of Theorem 1 we deduce that either $|V| < (4\lambda/(1-13\lambda))|S|$ or $|U| < ((2\lambda + 13\lambda^2)/(1-13\lambda) + \theta)|S|$.

In either case, $|V| < ((4\lambda + 13\lambda^2)/(1-13\lambda) + \theta)|S|$. Therefore,

$$|C| = |S| - |V| > \left(1 - \theta - \frac{4\lambda + 13\lambda^2}{1 - 13\lambda}\right)|S|,$$

as required.

Consider the following additional conditions on $(\mathcal{R}_i)_{i\geqq 1}$ and $(\mathcal{W}_i)_{i\geqq 0}$:

(a) There exists a constant $\eta > 0$ such that, for any $i \geqq 1$, $R \in \mathcal{R}_i$ and $k \geqq 1$,

$$R^k = Q \pmod{N_{i-1}}$$

implies $|R^k| < (1+\eta)|Q|$, i.e. R^k is almost (up to η) the shortest representative of its coset modulo N_{i-1}.

(b) The lengths of words in $\mathcal{W}_i$ are bounded by some constant w_i, $i \geqq 0$.

(c) Denote $\eta_j = \min\{|R| \mid R \in \mathcal{R}_j\}$, $j \geqq 1$. The constants λ, η, w_i, η_j satisfy the following inequalities:

$$\frac{4w_{i-1}}{\eta_i} < \frac{1-\eta}{1+\eta}, \quad \frac{4w_{i-1}}{\eta_i} + 2\,\frac{4\lambda + 13\lambda^2}{1-13\lambda} < \frac{1}{1+\eta} \qquad (i \geqq 1).$$

Corollary 2. *Let* $(\mathcal{R}_i)_{i \geqq 1}$ *and* $(\mathcal{W}_i)_{i \geqq 0}$ *satisfy* S(λ, θ), (L) *and the additional conditions* (a), (b), (c). *Then, in the notation of part* (i) *of Theorem* 1 *if* $R \equiv BU$ *then*

$$|Z_1^{-1}U^{-1}Z_2| < |A|, \quad \textit{hence} \quad |W_1Z_1^{-1}U^{-1}Z_2W_2| < |W|$$

and if $B \equiv R^mR'$ *then* $|Z_1^{-1}R'Z_2| < |A|$, *hence* $|W_1Z_1^{-1}R'Z_2^{-1}W_2| < |W|$.

Proof. Let $R \equiv BU$. Since $A^{-1}Z_1^{-1}BZ_2 \in N_{i-1}$, it follows that $R = Z_1AZ_2^{-1}U$ (mod N_{i-1}). Then, by (a), $|R| \leqq (1+\eta)|Z_1AZ_2^{-1}U|$. By (b), $|Z_1| \leqq w_{i-1}$ and, by (c), $|R| \geqq \eta_i$. Therefore, $|Z_1| < (w_{i-1}/\eta_i)|R|$. Similarly, $|Z_2| < (w_{i-1}/\eta_i)|R|$. By Corollary 1, $|U| < ((4\lambda + 13\lambda^2)/(1-13\lambda))|R|$. Then

$$|A| \geqq \left(\frac{1}{1+\eta} - \frac{4\lambda + 13\lambda^2}{1-13\lambda} - \frac{2w_{i-1}}{\eta_i}\right)|R|.$$

On the other hand, $|Z_1^{-1}U^{-1}Z_2| < (2w_{i-1}/\eta_i + (4\lambda + 13\lambda^2)/(1-13\lambda))|R|$. By (c),

$$\frac{2w_{i-1}}{\eta_i} + \frac{4\lambda + 13\lambda^2}{1-13\lambda} < \frac{1}{1+\eta} - \frac{2w_{i-1}}{\eta_i} - \frac{4\lambda + 13\lambda^2}{1-13\lambda}$$

and, therefore, $|Z_1^{-1}U^{-1}Z_2| < |A|$ and $|W_1Z_1^{-1}U^{-1}Z_2W_2| < |W|$.

Let $B \equiv R^mR'$. Since $A^{-1}Z_1^{-1}BZ_2 \in N_{i-1}$, it follows that

$$R^{m+1} = Z_1AZ_2^{-1}R'' \ (\text{mod } N_{i-1}).$$

Then, by (a), $(m+1)|R| < (1+\eta)|Z_1AZ_2^{-1}R''|$. We obtain

$$|A| \geqq \left(\frac{m+1}{1+\eta} - \frac{2w_{i-1}}{\eta_i} - \frac{|R''|}{|R|}\right)|R|.$$

We have $|Z_1^{-1}R'Z_2| \leqq (2w_{i-1}/\eta_i + |R'|/|R|)|R|$. Since $|R'| + |R''| = |R|$, by (c),

$$\frac{2w_{i-1}}{\eta_i} + \frac{|R'|}{|R|} \leqq \frac{1-\eta}{1+\eta} - \frac{2w_{i-1}}{\eta_i} + \left(1 - \frac{|R''|}{|R|}\right)$$

$$\leqq \frac{m-\eta}{1+\eta} + 1 - \frac{2w_{i-1}}{\eta_i} - \frac{|R''|}{|R|} = \frac{m+1}{1+\eta} - \frac{2w_{i-1}}{\eta_i} - \frac{|R''|}{|R|}.$$

Therefore, $|Z_1^{-1}R'Z_2| < |A|$ and $|W_1Z_1^{-1}R'Z_2W_2| < |W|$, as required.

If $R \equiv BU$, take $W' := W_1 Z_1^{-1} U^{-1} Z_2 W_2$ and if $B \equiv R^m R'$, take $W' : W_1 Z_1^{-1} R' Z_2 W_2$. We have $W' = W \pmod{N_i}$ and, by Corollary 2, $|W'| < |W|$. We use this fact to show that, under certain additional conditions, there is an analog of Dehn's Algorithm which solves the word problem in $G = F/N$.

THEOREM 2. *Let the set X of generators of F be countable, let $(\mathcal{R}_i)_{i \geqq 1}$ and $(\mathcal{W}_i)_{i \geqq 0}$ satisfy the conditions* S(λ, θ), (L), (a), (b), (c) *and, additionally*:

(d) *The $\mathcal{R}_i$ have a uniformly solvable word problem, i.e., there is a recursive procedure $\Phi(i, W)$ which, when given i and a word W, decides whether $W \in \mathcal{R}_i$.*

(e) *The elements of $\mathcal{R} = \bigcup_{i \geqq 1} \mathcal{R}_i$ of a fixed length are uniformly listable, i.e. there is a recursive procedure $\Psi(n)$ which, when given n, actually lists all words of $\mathcal{R}$ of length $\leqq n$ (together with the indices of the $\mathcal{R}_j$ to which they belong).*

(f) *The sets $\mathcal{W}_i$ are uniformly listable.*

The above hypotheses are sufficient for the effectiveness of applying Dehn's Algorithm to $G = F/N$.

(I am grateful to Professor P. E. Schupp who has corrected the statement of conditions (d), (e), (f) of Theorem 2 (communicated to me by Professor J. J. Rotman).)

We show now how Theorem 2 is deduced from Corollary 2 to Theorem 1.

For the moment, let us say that a word W in F is *i-reducible* if there exist a factorization $W \equiv W_1 A W_2$, two words $Z_1, Z_2 \in \mathcal{W}_{i-1}$, a word B and a relator $R \in \mathcal{R}_i$ such that $A^{-1} Z_1^{-1} B Z_2 \in N_{i-1}$ and either (1) $R \equiv BU$ and $|Z_1^{-1} U^{-1} Z_2| < |A|$, or (2) $B \equiv R^m R'$ with $m \geqq 1$, $R \equiv R' R''$ and $|Z_1^{-1} R' Z_2| < |A|$.

If a word is not i-reducible, we call it *i-reduced.*

In the first case $|Z_1 A Z_2^{-1} U| = |A| + |Z_1^{-1} U^{-1} Z_2| < 2|A|$. We have $R \equiv BU = Z_1 A Z_2^{-1} U \pmod{N_{i-1}}$; hence, by (a),

$$|R| \leqq (1+\eta)|Z_1 A Z_2^{-1} U| < 2(1+\eta)|A| \leqq 2(1+\eta)|W|.$$

In the second case $R^m = Z_1 A Z_2^{-1} R'^{-1} \pmod{N_{i-1}}$, hence

$$|R^m| \leqq (1+\eta)|Z_1 A Z_2^{-1} R'^{-1}| = (1+\eta)(|A| + |Z_1^{-1} R' Z_2|)$$
$$\leqq 2(1+\eta)|A| \leqq 2(1+\eta)|W|.$$

In view of (d) and (e), we can effectively list all words in $\mathcal{R}_i$ of length $< 2(1+\eta)|W|$. By (f), $\mathcal{W}_{i-1}$ is a finite set. Therefore, if F/N_{i-1} has a solvable word problem, we can effectively decide whether or not a given word W is i-reducible and in case W is i-reducible, we can effectively find a word W' such that $|W'| < |W|$ and $W' = W \pmod{N_i}$.

We can now show by induction on k that each F/N_k has a solvable word problem. This is clear for $k=0$, because $N_0 = E$ and $F/N_0 \cong F$. Let us assume that F/N_i has a solvable word problem for $i<k$.

Let W be a word in F. In view of the above remarks, we can effectively find a word W_0 such that $W = W_0 \pmod{N_k}$, $|W_0| \leqq |W|$ and W_0 is i-reduced for any $i \leqq k$.

If $W_0 \equiv 1$ then $W \in N_k$. If $W_0 \not\equiv 1$, then applying Corollary 2 to W_0 with $(\mathcal{R}_i)_{i\geqq 1}$ replaced by $(\mathcal{R}'_i)_{i\geqq 1}$ where $\mathcal{R}'_i = \mathcal{R}_i$ for $i \leqq k$ and $\mathcal{R}'_i = \emptyset$ for $i>k$, we obtain $W_0 \notin N_k$ and therefore $W \notin N_k$. Thus F/N_k has a solvable word problem.

We now turn to the word problem in F/N. Let W be a word in F. In view of (d) and (e), we can effectively find an integer h such that, for any $i>h$, the set $\mathcal{R}_i$ does not contain words of length $<2(1+\eta)|W|$. Since for any i, F/N_i has a solvable word problem, we can effectively find a word W_0 such that $|W_0| \leqq |W|$, $W = W_0 \pmod{N_h}$ and W_0 is i-reduced for any $i \leqq h$. We claim that W_0 is i-reduced for any $i>h$ as well. Indeed, if W_0 is i-reducible for some i then $\mathcal{R}_i$ contains a relator R such that $|R| < 2(1+\eta)|W_0| \leqq 2(1+\eta)|W|$. By our choice of h, this cannot happen for $i>h$. Thus, W_0 is i-reduced for all $i \geqq 1$.

$W = W_0 \pmod{N_h}$ implies $W = W_0 \pmod{N}$.

If $W_0 \equiv 1$ then $W \in N$. If $W_0 \not\equiv 1$ then, by Corollary 2, $W_0 \notin N$ and therefore $W \notin N$. Thus, F/N has a solvable word problem, as required.

1.5. In this paper we shall only develop the machinery, leaving the applications to subsequent papers of this series. For this reason, we should like to describe briefly a few examples that give an idea of how the method works. Most of these examples are known even in a stronger form. They will not be used in the rest of the paper.

1°. *Ordinary small cancellation.* Consider the case in which all the sets $\mathcal{R}_i$, except $\mathcal{R}_1$, are empty and $\mathcal{W}_0 = \{1\}$.

Then, for $R \equiv U_1 A U_2 \in \mathcal{R}_1$, A is a 1-piece of R if and only if there is a relator $S \equiv AV \in \mathcal{R}_1$ such that $V^{-1}U_2U_1$ is not freely equal to 1 or to a conjugate of some relator $T \in \mathcal{R}_1$.

Condition (S) now asserts that no relator can be written as a product of less than 9 1-pieces. Since all the sets $\mathcal{R}_i$, $i>1$, are empty, we can enlarge the sets $\mathcal{W}_i$, $i>0$, without affecting condition (S). For example, we can take $\mathcal{W}_i = F$ for $i>0$. Then condition (L) is automatically satisfied.

Theorem 1 asserts that every freely reduced word W in N contains a subword A such that, for some $R \in \mathcal{R}_1$, we have $R \equiv AQ_1Q_2Q_3Q_4$, where the Q_i's are 1-pieces of R.

2°. *Small cancellation in free products.* Let $H = \Pi_\alpha^* H_\alpha$, a free product of groups. For a subset $\mathcal{S}$ of H, let K be the normal subgroup of H generated by $\mathcal{S}$ and let $G = H/K$.

For each H_α, we have $H_\alpha = F_\alpha / U_\alpha$ for some free group F_α and normal subgroup U_α of F_α. Let $F = \Pi_\alpha^* F_\alpha$ and let $\sigma : F \to H$ be induced by epimorphisms $F_\alpha \to H_\alpha$.

Consider the following families of sets $(\mathcal{R}_i)_{i \geqq 1}$ and $(\mathcal{W}_i)_{i \geqq 0}$ in F. Let $\mathcal{V}_\alpha$ be the subset of U_α consisting of all non-trivial cyclically reduced words. Take $\mathcal{R}_1 = \bigcup_\alpha \mathcal{V}_\alpha$. Let $\mathcal{R}_2'$ be a subset of F such that $\sigma(\mathcal{R}_2') = \mathcal{S}$ and $\mathcal{R}_2$ the symmetrized closure of $\mathcal{R}_2'$. Let $\mathcal{R}_i = \emptyset$ for $i > 2$. Put $\mathcal{W}_0 = \{1\}$, $\mathcal{W}_1 = \{1\}$, and $\mathcal{W}_i = F$ for $i \geqq 2$.

Then $F/N_1 \cong H$ and $F/N_2 \cong G$, where N_i denotes the normal subgroup of F generated by $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_i$.

Applying Theorem 1 to $(\mathcal{R}_i)_{i \geqq 1}$ and $(\mathcal{W}_i)_{i \geqq 0}$, we obtain a small-cancellation theorem for free products of groups. However, its hypotheses are more restrictive and its conclusion is weaker than in the known results (see, for example, [1], p. 278), so we shall not go into details.

3°. *Small cancellation in HNN-extensions.* Let H be a group, P and Q subgroups of H and $\phi : P \to Q$ an isomorphism. Let

$$L = \langle H, t \mid t^{-1} a t = \phi(a) \text{ for } a \in P \rangle$$

be the corresponding HNN-extension. Let $\mathcal{S}$ be a subset of L, let K be the normal subgroup of L generated by $\mathcal{S}$ and let $G = L/K$.

We have $H \cong F_0/U$ for some free group F_0 and a normal subgroup U of F_0. Let $F = F_0 * \langle t \rangle$ and let $\rho : F \to L$ be the extension of $F_0 \to H$ determined by $\rho(t) = t$.

Consider the following families of sets $(\mathcal{R}_i)_{i \geqq 1}$ and $(\mathcal{W}_i)_{i \geqq 0}$ in F. Let $\mathcal{R}_1$ be the subset of U consisting of all non-trivial cyclically reduced words. $\mathcal{R}_2$ is the symmetrized closure of the set of words

$$\{t^{-1} V_1 t V_2^{-1} \mid V_1, V_2 \in F_0, \rho(V_1) \in P, \rho(V_2) \in Q, \phi\rho(V_1) = \rho(V_2)\}.$$

Let $\mathcal{R}_3'$ be a subset of F such that $\rho(\mathcal{R}_3') = \mathcal{S}$, and $\mathcal{R}_3$ the symmetrized closure of $\mathcal{R}_3'$. Take $\mathcal{R}_i = \emptyset$ for $i > 3$, $\mathcal{W}_0 = \{1\}$, $\mathcal{W}_1 = \{1\}$, $\mathcal{W}_2 = F_0$ and $\mathcal{W}_i = F$ for $i \geqq 3$.

Then $F/N_1 \cong H * \langle t \rangle$, $F/N_2 \cong L$ and $F/N_3 \cong G$. Applying Theorem 1 to $(\mathcal{R}_i)_{i \geqq 1}$ and $(\mathcal{W}_i)_{i \geqq 0}$, we obtain a small cancellation theorem for HNN-extensions of groups, which is, however, considerably weaker than the known results (see, for example, [1], p. 292).

4°. Let F be a free group on free generators x, y. Let $U_1, U_2, \cdots$ be a sequence of words in F and $n_1, n_2, \cdots$ a sequence of positive integers. Let $k_1, k_2, \cdots$ and $l_1, l_2, \cdots$ be two sequences of positive integers such that $k_i < l_i$ for all $i \geqq 1$.

We define words $V_1, V_2, \cdots$ and sets of words $\mathcal{R}_1, \mathcal{R}_2, \cdots$ inductively, as follows:

Suppose that $V_1, V_2, \cdots, V_{i-1}$ and $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_{i-1}$ have already been defined. Let V_i be a shortest possible word such that $U_i = L_i^{-1} V_i L_i \pmod{N_{i-1}}$ for some L_1. Let

$$T_i := xy^{k_i+1}xy^{k_i+2}\cdots xy^{l_i}.$$

Define the set $\mathcal{R}'_i$ by

$$\mathcal{R}'_i := \left\{ T_i^{p_1}V_i^{q_1}T_i^{p_2}V_i^{q_2}\cdots T_i^{p_r}V_i^{q_r} \,\middle|\, \sum_{j=1}^{r} p_j + n_i q_j = 0,\ r = 1, 2, \cdots \right\}.$$

To each $R' \in \mathcal{R}'_i$ we assign a reduced cyclically reduced word R'' such that R'' is freely equal to $P^{-1}R'P$ for some P. Let $\mathcal{R}''_i := \{R'' \mid R' \in \mathcal{R}'_i\}$ and let $\mathcal{R}_i$ be the symmetrized closure of $\mathcal{R}''_i$.

In a subsequent paper we intend to show that, if the sequences $k_1, k_2, \cdots$ and $l_1, l_2, \cdots$ and the sets $\mathcal{W}_i$, $i \geqq 0$, are suitably chosen then conditions (S) and (L) are satisfied.

Then part (ii) of Theorem 1 implies that $N \neq F$ and therefore the group $G = G/N$ is non-trivial. On the other hand, it is immediate from the construction of G that T_iN is the n_i-th root of V_iN in G since $\mathcal{R}'_i$ contains the word $T_i^{n_i}V_i^{-1}$. Then $(L_i^{-1}T_iL_i)N$ is the n_i-th roof of U_iN in G.

Therefore, for a suitable choice of the sequences $U_1, U_2, \cdots$ and $n_1, n_2, \cdots$, G will be a finitely generated non-trivial divisible group.

§2. Van Kampen diagrams and restatement of the main results

2.1. Definition 2. *Maps in the plane.* Let $\mathbf{E}^2$ denote the Euclidean plane. We shall consider only piecewise linear subsets of $\mathbf{E}^2$. If $S \subseteq \mathbf{E}^2$, then bd(S) will denote the boundary of S; the topological closure of S will be denoted by clos(S) and the interior of S by int(S). compl(S) will denote $\mathbf{E}^2 \setminus S$.

A *vertex* is a point of $\mathbf{E}^2$. An *edge* is a bounded subset of $\mathbf{E}^2$ homeomorphic to the open unit interval. A *region* is a bounded set homeomorphic to the open unit square. A *map* M is a finite collection of vertices, edges and regions which are pairwise disjoint and satisfy the following conditions:

(i) If e is an edge of M, there are vertices P and Q (not necessary distinct) such that clos$(e) = e \cup \{P\} \cup \{Q\}$.

(ii) If Φ is a region of M, there are edges $e_1, e_2, \cdots, e_n$ in M such that $\mathrm{bd}(\Phi) = \mathrm{clos}(e_1) \cup \cdots \cup \mathrm{clos}(e_n)$.

An example of a map is shown in Fig. 4.

The *support*, $\mathrm{supp}(M)$, of M is the set theoretic union of all its vertices, edges and regions. We write $\mathrm{bd}(M)$ instead of $\mathrm{bd}(\mathrm{supp}(M))$ and so on. Let $\mathrm{Reg}(M)$ denote the set of regions of M.

DEFINITION 3. *Paths.* Every edge of M can be oriented in either of two directions. If e is an oriented edge, we denote by $\mathrm{o}(e)$ the *initial vertex* of e and by $\mathrm{t}(e)$ the *terminal vertex* of e. A *path* $\mu = (v_0, e_1, v_1, e_2, \cdots, e_m, v_m)$ is a sequence of vertices v_i and oriented edges e_i such that $\mathrm{o}(e_j) = v_{j-1}$ and $\mathrm{t}(e_j) = v_j$, $1 \leqq j \leqq m$. We use the notation $\mathrm{o}(\mu) = v_0$ and $\mathrm{t}(\mu) = v_m$ for the initial and terminal vertices of μ. We identify a *trivial path* (v) with the corresponding vertex v. If $\mathrm{o}(\mu) = \mathrm{t}(\mu)$ we call μ a *closed path* or a *cycle.* The path $\mu^{-1} = (v_m, e_m^{-1}, \cdots, e_2^{-1}, v_1, e_1^{-1}, v_0)$ is called the *inverse* of μ, where e^{-1} denotes the edge e with the inverse orientation. The number m is called the *length* of μ. We denote $|\mu| := m$ (":=" means "equal by definition").

If $0 \leqq r \leqq s \leqq m$, the path $\nu = (v_r, r_{r+1}, \cdots, e_s, v_s)$ is called a *subpath* of μ. If $r = 0$ we say that ν is a *head* of μ, and if $s = m$ we say that ν is a *tail* of μ. A path μ is said to be *reduced* if it does not contain subpaths of the form (v, e, v', e^{-1}, v). If λ and μ are *paths* and $\mathrm{t}(\lambda) = \mathrm{o}(\mu)$ then the *product* $\lambda\mu$ is defined in the obvious sense. We call the path μ *simple* if $v_i \neq v_j$ for $i \neq j$.

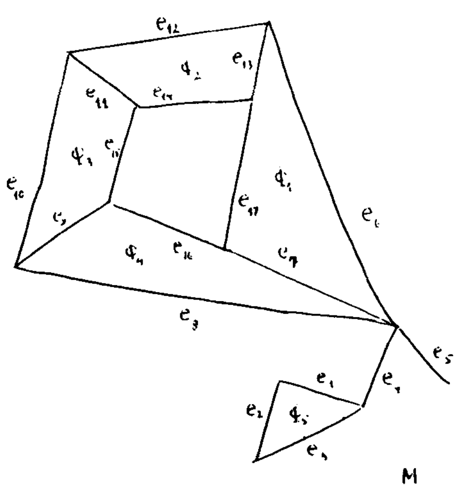


Fig. 4.

DEFINITION 4. *Boundary cycles and paths.* If Φ is a region of M, then a *boundary cycle* of Φ is a cycle α of minimal length which contains all the edges of $\mathrm{bd}(\Phi)$ and which does not cross itself in the sense defined in [1], p. 236.

For example, in Fig. 5 the path

$$\alpha = (v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_3^{-1}, v_2, e_4, v_4, e_4^{-1}, v_2, e_5, v_0)$$

does not cross itself and therefore is a boundary cycle of Φ, while

$$\beta = (v_0, e_1, v_1, e_2, v_2, e_4, v_4, e_4^{-1}, v_2, e_3, v_3, e_3^{-1}, v_2, e_5, v_0)$$

crosses itself and therefore is not a boundary cycle.

In a similar way we define a boundary cycle of a connected component of M and a boundary cycle of a connected component of the complement to $\mathrm{supp}(M)$.

Let α be a cycle and n an integer. We define α^n as follows:

(1) α^0 is the trivial path $o(\alpha)$;

(2) $\alpha^n := \alpha\alpha^{n-1}$ for $n > 0$;

(3) $\alpha^n := (\alpha^{-1})^{-n}$ for $n < 0$.

We call α^n the *n-th power* of α.

A *boundary path* of a region Φ is a subpath of a power of a boundary cycle of Φ.

Thus, for example, according to our definition, in Fig. 5, $\gamma = (v_1, e_2, v_2, e_4, v_4)$ is not a boundary path of Φ.

DEFINITION 5. *Normalized maps.* A map M is said to be *normalized* if none of its regions has vertices of degree 1 on its boundary.

For example, the map M_1 in Fig. 6 is not normalized, while the map M_2 is normalized.

Throughout this paper we shall consider *only* normalized maps whenever a new map is constructed we shall verify that it is normalized.

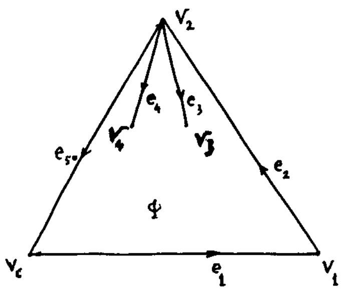


Fig. 5.

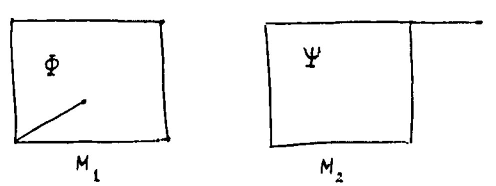


Fig. 6.

It is easily seen that in a normalized map each boundary path is reduced.

Definition 6. *Regular maps.* A map is said to be *regular* if each of its edges is on the boundary of some region.

Thus, for example, the map M_1 in Fig. 7 is not regular, while the map M_2 is regular.

It is obvious that a regular submap of a given map is uniquely determined by the set of its regions.

Let μ and ν be two paths in a map M such that $o(\mu)=o(\nu)$ and $t(\mu)=t(\nu)$. In an obvious way we define the notion "*μ is homotopic to ν in M*".

2.2. *Sets of paths in ranked maps.*

Definition 7. *Ranked maps.* A ranked map $\mathcal{M}=(M,\text{rank})$ is a map M equipped with a function

$$\text{rank}:\text{Reg}(M)\rightarrow\{1,2,\cdots\}.$$

Definition 8. *Equivalence of paths in a ranked map.* Let $\mathcal{M}=(M,\text{rank})$ be a ranked map. Let μ and ν be two paths in M such that $o(\mu)=o(\nu)$ and $t(\mu)=t(\nu)$. Let $i>0$. We say that μ and ν are *i-equivalent,* writing $\mu\sim_i\nu$, if μ is homotopic to ν in the map M_i obtained from M by deleting all regions Φ of rank $>i$.

Definition 9. *Sets of paths* $\text{Br}(k)$, $\mathcal{P}(\Phi;j)$, $\mathcal{I}(\Phi;c)$, $\mathcal{H}(\Phi;c)$. Let $\mathcal{M}=(M,\text{rank})$ be a ranked map. Let Φ be a region in M of rank k. Let $j\geqq 1$ be an integer, and let $d^s=(d_1^s,d_2^s,\cdots)$, $c=(c_1,c_2,\cdots)$ be sequences of numbers, $s=1,2,\cdots$.

We define sets of paths $\text{Br}_{\mathcal{M}}^{(d^s)}(0)$, $\text{Br}_{\mathcal{M}}^{(d^s)}(k)$, $\mathcal{P}_{\mathcal{M}}^{(d^s)}(\Phi;j)$, $\mathcal{I}_{\mathcal{M}}^{(d^s)}(\Phi;c)$, $\mathcal{H}_{\mathcal{M}}^{(d^s)}(\Phi;c)$ for $k=1,2,\cdots$ inductively, as follows.

(1) The set $\text{Br}_{\mathcal{M}}^{(d^s)}(0)$ consists of all trivial paths in M.

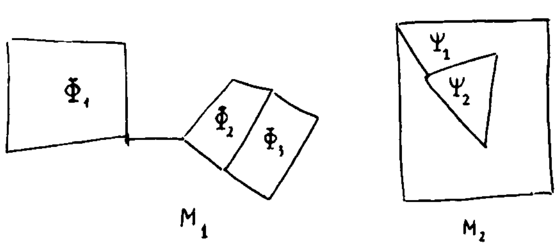


Fig. 7.

Now let us assume that the sets $\mathrm{Br}_{\mathcal{M}}^{(d^s)}(h)$ for $h = 0, 1, \cdots, k-1$ have already been defined.

(2) A path μ in M belongs to $\mathcal{P}_{\mathcal{M}}^{(d^s)}(\Phi; j)$ if and only if

(α) μ is a boundary path of Φ

and there exist

(β) simple paths $\sigma, \tau \in \mathrm{Br}_{\mathcal{M}}^{(d^s)}(h)$, where $h = \min(k, j) - 1$ (recall that rank $(\Phi) = k$);

(γ) a region $\Psi, \Psi \neq \Phi$, of rank j,

(δ) a boundary path ν of Ψ such that

(ε) $\mu \sim_h \sigma\nu\tau^{-1}$ (see Fig. 8).

(3) A path ξ in M belongs to $\mathcal{S}_{\mathcal{M}}^{(d^s)}(\Phi; c)$ if and only if

(α) ξ is a boundary path of Φ;

(β) there is a factorization $\xi = \xi_1 \xi_2 \cdots \xi_m$ where each ξ_e belongs to $\mathcal{P}_{\mathcal{M}}^{(d^s)}(\Phi; f(e))$ for some $f(e)$;

(γ) $\mathrm{card}\{e \mid f(e) = i\} \leqq c_i$ $(i = 1, 2, \cdots)$, where $c = (c_1, c_2, \cdots)$ (see Fig. 9).

(4) A path η in M belongs to $\mathcal{H}_{\mathcal{M}}^{(d^s)}(\Phi; c)$ if and only if every subpath η_0 of η belongs to $\mathcal{S}_{\mathcal{M}}^{(d^s)}(\Phi; c)$.

(5) A path ν in M belongs to $\mathrm{Br}_{\mathcal{M}}^{(d^s)}(k)$ if and only if $\nu = \nu_1 \sigma \nu_2$, where $\nu_1, \nu_2 \in \mathrm{Br}_{\mathcal{M}}^{(d^s)}(k-1)$ and either σ is trivial or σ is a boundary path of some region Φ of rank k such that

(α) σ does not contain a boundary cycle of Φ;

(β) $\sigma \in \mathcal{H}_{\mathcal{M}}^{(d^s)}(\Phi; d^k)$ (see Fig. 10) (note that this is the only point where the dependence on the d^s's actually appears!).

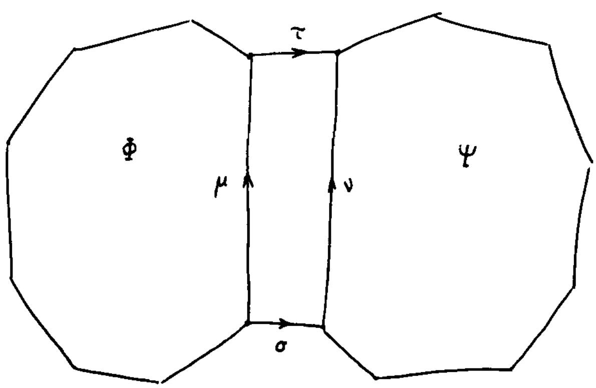


Fig. 8.

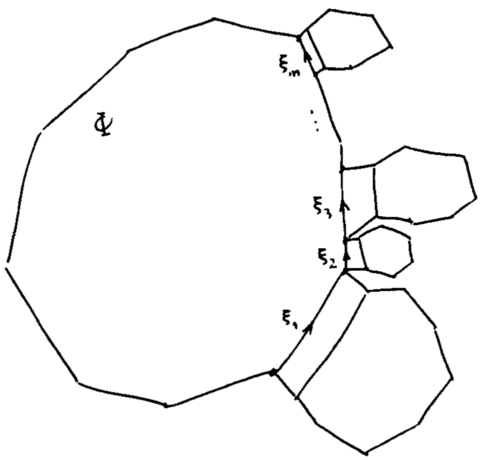


Fig. 9.

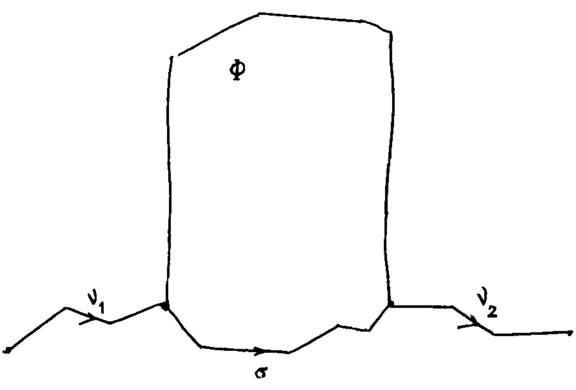


Fig. 10.

In the sequel we shall *fix* the sequences d^s as follows:

$$(\mathrm{L}_0) \qquad d^s = \sum_{h=1}^{s-1} 2\cdot 13^{s-h} e_h + e_s, \qquad s = 1, 2, \cdots$$

and omit the upper index (d^s). We shall also omit the lower index $\mathcal{M}$ whenever it is clear from the contex to which ranked map we are referring. We thus write $\mathrm{Br}(k)$ instead of $\mathrm{Br}_{\mathcal{M}}^{(d^s)}(k)$, and so on.

In the following two lemmas we collect some properties of the sets of paths defined above which we need later on.

Lemma 1. (a) *The sets of paths* $\mathrm{Br}(k)$, $\mathcal{P}(\Phi; j)$, $\mathcal{I}(\Phi; c)$ *and* $\mathcal{H}(\Phi; c)$ *are closed under taking inverse paths.*

(b) *$\mathcal{H}(\Phi; c)$ is closed under passage to subpaths.*

(c) $\mathrm{Br}(k-1) \subseteq \mathrm{Br}(k)$ *for* $k \geqq 1$.

(d) $\mathrm{Br}(k)$ *is closed under passage to subpaths.*

(e) *Let* $\mathcal{N} = (N, \mathrm{rank})$ *be a ranked map such that* N *is a submap of* M *and for any* $\Phi \in \mathrm{Reg}(N)$, $\mathrm{rank}_{\mathcal{M}}(\Phi) = \mathrm{rank}_{\mathcal{N}}(\Phi)$. *Then* $\mathrm{Br}_{\mathcal{N}}(k) \subseteq \mathrm{Br}_{\mathcal{M}}(k)$ *and, for any* $\Phi \in \mathrm{Reg}(N)$, $\mathcal{P}_{\mathcal{N}}(\Phi; j) \subseteq \mathcal{P}_{\mathcal{M}}(\Phi; j)$, $\mathcal{I}_{\mathcal{N}}(\Phi; c) \subseteq \mathcal{I}_{\mathcal{M}}(\Phi; c)$, $\mathcal{H}_{\mathcal{N}}(\Phi; c) \subseteq \mathcal{H}_{\mathcal{M}}(\Phi; c)$.

PROOF. Parts (a), (b) and (c) are obvious. Since $\lambda \sim_i \mu$ in $\mathcal{N}$ implies $\lambda \sim_i \mu$ in $\mathcal{M}$, part (e) follows by induction on k and rank(Φ). Let us prove part (d). For $k = 0$, Br(0) consists of trivial paths and the assertion is obvious. Let $k > 0$. Let $\nu \in \mathrm{Br}(k)$. Then $\nu = \nu_1 \sigma \nu_2$, where $\nu_1, \nu_2 \in \mathrm{Br}(k-1)$ and σ, if non-trivial, is a boundary path of some region Φ of rank k satisfying conditions (5) (α), (β) of Definition 9. If τ is a subpath of ν then there exists a factorization

$$\tau = \tau_1 \rho \tau_2,$$

where $\tau_1(\rho_1, \tau_2)$, if non-trivial, is a subpath of $\nu_1(\sigma, \nu_2)$; hence $\tau_1, \tau_2 \in \mathrm{Br}(k-1)$ by the induction hypothesis and ρ, if non-trivial, is a boundary path of Φ such that ρ does not contain a boundary cycle of Φ. By part (b), (5), (β) implies $\rho \in \mathcal{H}(\Phi; \sum_{j=1}^{k-1} 2 \cdot 13^{k-j} e_j + e_k)$ (recall that, by (L_0), $d^k = \sum_{j=1}^{k-1} 2 \cdot 13^{k-j} e_j + e_k$). Therefore $\tau \in \mathrm{Br}(k)$. This proves the lemma.

LEMMA 2. *Let* $l \geqq 0$ *and assume that, for any region* Π *in* M *of rank* $\leqq l$, clos(Π) *is simply-connected. If for* $\mu \in \mathrm{Br}(l)$ *there exists a factorization* $\mu = \mu_1 \mu_2 \mu_3$ *such that* $\mathrm{t}(\mu_1) = \mathrm{o}(\mu_3)$ *then* $\mu_1 \mu_3 \in \mathrm{Br}(l)$.

PROOF. We proceed by induction on l. If $l = 0$, then μ is a trivial path and there is nothing to prove. Let $l > 0$. We have $\mu = \nu_1 \sigma \nu_2$ where $\nu_1, \nu_2 \in \mathrm{Br}(l-1)$ and σ, if non-trivial, is a boundary path of some region Ψ of rank l such that σ does not contain a boundary cycle of Ψ and $\sigma \in \mathcal{H}(\Psi; \sum_{j=1}^{l-1} 2 \cdot 13^{l-j} e_j + e_l)$. We have to consider several different possibilities.

Case 1. $\mu_1 \mu_2$ is a head of ν_1.

Then for some τ_1, $\nu_1 = \mu_1 \mu_2 \tau_1$ and $\mu_3 = \tau_1 \sigma \nu_2$. We have $\mathrm{t}(\mu_1) = \mathrm{o}(\mu_3) = \mathrm{o}(\tau_1)$ and then, by the induction hypothesis, $\mu_1 \tau_1 \in \mathrm{Br}(l-1)$; hence $\mu_1 \mu_3 = \mu_1 \tau_1 \sigma \nu_2 \in \mathrm{Br}(l)$.

Case 2. $\mu_2 \mu_3$ is a tail of ν_2.

Then for some τ_2, $\nu_2 = \tau_2 \mu_2 \mu_3$ and $\mu_1 = \nu_1 \sigma \tau_2$. As in the previous case, we obtain $\tau_3 \mu_3 \in \mathrm{Br}(l-1)$ and $\mu_1 \mu_3 = \nu_1 \sigma \tau_2 \mu_3 \in \mathrm{Br}(l)$.

Case 3. μ_1 is a head of ν_1 and μ_3 is a tail of ν_2. By Lemma 1(d), $\mu_1, \mu_3 \in \mathrm{Br}(l-1)$, hence $\mu_1\mu_3 \in \mathrm{Br}(l)$.

Case 4. μ_1 is a head of ν_1, ν_1 is a head of $\mu_1\mu_2$ and ν_2 is a tail of μ_3.

Then for some τ_3, τ_4 and τ_5, $\nu_1 = \mu_1\tau_3$, $\sigma = \tau_4\tau_5$, $\mu_3 = \tau_5\nu_2$. By Lemma 1(d), $\mu_1 \in \mathrm{Br}(l-1)$ and by Lemma 1(b), if τ_5 is non-trivial then $\tau_5 \in \mathcal{H}(\Psi; \sum_{j=1}^{l-1} 2\cdot 13^{l-j}e_j + e_l)$. Then $\mu_1\mu_3 = \mu_1\tau_5\nu_2 \in \mathrm{Br}(l)$.

Case 5. ν_1 is a head of μ_1, μ_3 is a tail of ν_2 and ν_2 is a tail of $\mu_2\mu_3$.

This case is similar to Case 4.

Case 6. ν_1 is a head of μ_1 and ν_2 is a tail of μ_3.

Then for some τ_6, τ_7, $\mu_1 = \nu_1\tau_6$, $\sigma = \tau_6\mu_2\tau_7$, $\mu_3 = \tau_7\nu_2$. Let us show that μ_2 is trivial. Indeed, if μ_2 is non-trivial, then σ is non-trivial. Then since $o(\mu_2) = t(\mu_1) = o(\mu_3) = t(\mu_2)$, μ_2 is a closed boundary path of Ψ. Since $\mathrm{clos}(\Psi)$ is simply-connected, every non-trivial closed boundary path of Ψ contains a boundary cycle of Ψ, a contradiction. Hence μ_2 is trivial and then $\mu_1\mu_3 = \mu_1\mu_2\mu_3 \in \mathrm{Br}(l)$.

All the possibilities have been exhausted. The lemma is proved.

2.3. *Van Kampen diagrams.*

Definition 10. A van Kampen diagram over a group G is a map M and a function L assigning to each oriented edge e of M, as a *label*, an element $L(e)$ of G such that $L(e^{-1}) = L(e)^{-1}$.

We shall consider only van Kampen diagrams over free groups. We always assume that the label of each oriented edge is a generator or the inverse of a generator (from a fixed set of generators).

If $\mu = (v_0, e_1, v_1, \cdots, v_{m-1}, e_m, v_m)$ is a path in M, we define $L(\mu) := L(e_1)L(e_2)\cdots L(e_m)$.

Let $\mathcal{S}$ be a symmetrized subset of F. A van Kampen diagram is called an *$\mathcal{S}$-diagram* if, for any boundary cycle μ of any region Φ in M, we have $L(\mu) \in \mathcal{S}$.

The application of van Kampen diagram is based on the following lemma ([1], p. 237).

Lemma 3. *Let U be the normal subgroup of F generated by $\mathcal{S}$. A non-empty reduced word W belongs to U if and only if there is a connected simply-connected $\mathcal{S}$-diagram such that for some boundary cycle α of the underlying map we have $L(\alpha) \equiv W$.*

Let $(\mathcal{R}_i)_{i\geqq 1}$ be a family of disjoint symmetrized subsets of F; let $\mathcal{R} = \bigcup_{i\geqq 1}\mathcal{R}_i$ and let N be the normal subgroup of F generated by $\mathcal{R}$. Let (M, L) be an $\mathcal{R}$-diagram. We define the *rank* of a region Φ in M as follows:

rank$(\Phi) = i$ if and only if, for some boundary cycle ρ of Φ, $L(\rho)\in\mathcal{R}_i$. Since $\mathcal{R}_i$ is symmetrized we then have $L(\rho')\in\mathcal{R}_i$ for each boundary cycle ρ' of Φ. We obtain thus a ranked map $\mathcal{M} = (M, \text{rank})$.

Definition 11. *Minimal $\mathcal{R}$-diagrams.* For a ranked map $\mathcal{M} = (M, \text{rank})$ we define the *generating polynomial* $\text{gen}(\mathcal{M}) = \Sigma_{i\geqq 1} a_i t^i \in \mathbf{Z}[t]$, where a_i is the number of regions of M of rank i.

We introduce a nonarchimedian order on the ring of polynomials $\mathbf{Z}[t]$, taking $n < t$ for all $n\in\mathbf{Z}$.

Let W be a non-trivial reduced word in N and let (M, L) be a connected simply-connected $\mathcal{R}$-diagram such that $L(\alpha)\equiv W$ for some boundary cycle α of M. Then we call (M, L) an $\mathcal{R}$-diagram for W. Let $\mathcal{M} = (M, \text{rank})$ be the corresponding ranked map. We say that (M, L) is a *minimal $\mathcal{R}$-diagram for W* if, given any other $\mathcal{R}$-diagram (M_0, L) for W with the corresponding ranked map $\mathcal{M}_0 = (M_0, \text{rank})$, we have $\text{gen}(\mathcal{M})\leqq\text{gen}(\mathcal{M}_0)$.

For a minimal $\mathcal{R}$-diagram, there is a close connection between the sets of paths introduced in Definition 9 and the sets of words introduced in Definition 1. We have

Lemma 4. *Let $(\mathcal{R}_i)_{i\geqq 1}$ be family of disjoint symmetrized subsets of the free group F and let $(\mathcal{W}_i)_{i\geqq 0}$ be a family of subsets of F satisfying condition* (L). *Let W be a non-trivial reduced word in $N = \langle\bigcup_{i\geqq 1}\mathcal{R}_i\rangle^F$ and (M, L) a minimal $\bigcup_{i\geqq 1} R_i$-diagram for W. Let Φ be a region in M of rank $k\geqq 1$, ρ a boundary cycle of Φ and μ a subpath of ρ.*

(a) *If $\mu\in\mathcal{P}(\Phi; j)$ then $L(\mu)\in\mathcal{P}(L(\rho); j)$, $j\geqq 1$.*

(b) *If $\mu\in\mathcal{I}(\Phi; c)$ then $L(\mu)\in\mathcal{I}(L(\rho); c)$.*

(c) *If $\mu\in\mathcal{H}(\Phi; c)$ then $L(\mu)\in\mathcal{H}(L(\rho); c)$.*

(d) *If $\mu\in\text{Br}(k)$ then $L(\mu)\in\mathcal{W}_k$.*

Proof. We proceed by induction on k. If $k = 0$ then parts (a), (b), (c) are vacuous. If $\mu\in\text{Br}(0)$ then μ is a trivial path and then $L(\mu)\equiv 1\in\mathcal{W}_0$, by (L). Therefore, part (d) holds for $k = 0$. Let $k > 0$. We start with part (a). Let $\mu\in\mathcal{P}(\Phi; j)$. Then, according to Definition 9, there exist paths ν, σ, τ and a region Ψ of rank j satisfying (α), (β), (γ), (δ), (ε) of Definition 9, (2).

Since μ is a subpath of ρ, we have $\rho = \rho_1\mu\rho_2$ for some paths ρ_1, ρ_2. Then

$L(\rho)\equiv L(\rho_1)L(\mu)L(\rho_2)$. Since Φ is of rank k, we have $L(\rho)\in\mathcal{R}_k$. Thus, $L(\mu)$ is a subword of a relator $L(\rho)\in\mathcal{R}_k$.

Since, by (δ), ν is a boundary path of Ψ, there is a boundary cycle ω of Ψ such that, for some $m\geqq 1$, $\omega^m=\nu\omega'$. By (γ), rank$(\Psi)=j$, therefore $L(\omega)\in\mathcal{R}_j$ and $L(\nu)$ is an (initial) subword of $L(\omega)^m$. By (β), $\sigma,\tau\in\mathrm{Br}(h)$ where $h=\min(k,j)-1$. Then, by the induction hypothesis, $L(\sigma),L(\tau)\in\mathcal{W}_h$.

It is an immediate consequence of Definition 8 that if ξ_1 and ξ_2 are two paths in M such that $\xi_1\sim_i\xi_2$ then $L(\xi_1)=L(\xi_2)\ (\mathrm{mod}\ N_i)$, where N_i is the normal subgroup of F generated by $\mathcal{R}_1\cup\mathcal{R}_2\cup\cdots\cup\mathcal{R}_i$. Therefore (ε) implies

$$L(\mu)=L(\sigma)L(\nu)L(\tau)^{-1}\ (\mathrm{mod}\ N_h).$$

Now let us assume that $k=j$. We shall show that if the word

$$L(\sigma)L(\omega)^{-1}L(\sigma)^{-1}L(\mu)L(\rho_2)L(\rho_1)$$

(after reducing) belongs to $N_h=N_{k-1}$ or is conjugate modulo N_h to a relator $T\in\mathcal{R}_k$ then the $\mathcal{R}$-diagram (M,L) for W is not minimal, in a contradiction with our assumption.

In both cases we can construct an $\mathcal{R}$-diagram (M_0,L) for $L(\sigma)L(\omega)^{-1}L(\sigma)^{-1}L(\mu)L(\rho_2)L(\rho_1)$ such that for the corresponding ranked map $\mathcal{M}_0=(\mathcal{M}_0,\mathrm{rank})$ we have gen$(\mathcal{M}_0)<2t^k$.

Since σ is simple and $\Phi\neq\Psi$ (see (β) and (γ)), making a cut through σ and deleting the regions Φ and Ψ we obtain an $\mathcal{R}$-diagram $\tilde{M}$ (see Figs. 11, 12).

The boundary cycle of the hole is $\sigma'\omega^{-1}\sigma''^{-1}\mu\rho_2\rho_1$, where $L(\sigma')\equiv L(\sigma'')\equiv L(\sigma)$. We can therefore "fill in" the hole by the $\mathcal{R}$-diagram (M_0,L), obtaining a new $\mathcal{R}$-diagram (M_1,L) for W. Let $\mathcal{M}_1=(M_1,\mathrm{rank})$ be the corresponding ranked map. It is clear that

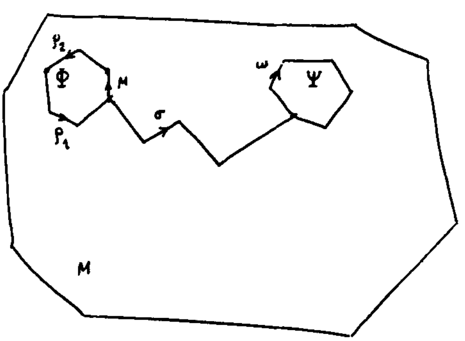


Fig. 11.

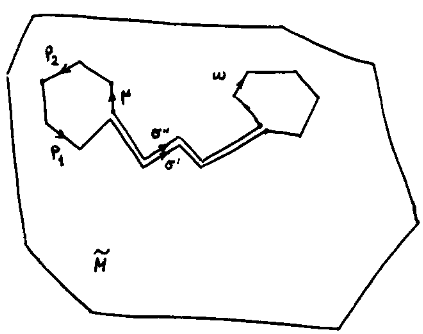


Fig. 12.

$$\mathrm{gen}(\mathcal{M}_1) = \mathrm{gen}(\mathcal{M}) - 2t^k + \mathrm{gen}(\mathcal{M}_0) < \mathrm{gen}(\mathcal{M})$$

which contradicts the minimality of (M, L). We have shown that all the requirements of Definition 1 with $A, U_1, U_2, R, B, S, Z_1, Z_2$ substituted by $L(\mu)$, $L(\rho_1)$, $L(\rho_2)$, $L(\rho)$, $L(\nu)$, $L(\omega)$, $L(\sigma)$, $L(\tau)$ respectively, are satisfied. Therefore $L(\mu) \in \mathcal{P}(L(\rho); j)$, i.e. $L(\mu)$ is a j-piece of $L(\rho)$. Part (a) of the lemma is proved. Parts (b) and (c) immediately follow from Definition 1 and Definition 9.

Now let $\mu \in \mathrm{Br}(k)$. Then $\mu = \mu_1 \sigma \mu_2$, where $\mu_1, \mu_2 \in \mathrm{Br}(k-1)$ and σ, if non-trivial, is a boundary path of some region Ψ of rank k such that σ does not contain a boundary cycle of Ψ and $\sigma \in \mathcal{H}(\Psi; \Sigma_{j=1}^{k-1} 2 \cdot 13^{k-j} e_j + e_k)$.

By the induction hypothesis, $L(\mu_1) \in \mathcal{W}_{k-1}$ and $L(\mu_2) \in \mathcal{W}_{k-1}$. If σ is trivial then, by condition (L),

$$L(\mu) \equiv L(\mu_1) L(\mu_2) \in \mathcal{W}_k.$$

Let σ be non-trivial. Then σ is a subpath of some boundary cycle ω of Ψ. By part (c),

$$L(\sigma) \in \mathcal{H}\left(L(\omega); \sum_{j=1}^{k-1} 2 \cdot 13^{k-j} e_j + e_k\right).$$

Applying condition (L) we obtain

$$L(\mu) \equiv L(\mu_1) L(\sigma) L(\mu_2) \in \mathcal{W}_k$$

because $L(\omega) \in \mathcal{R}_k$. This proves part (d).

The lemma is proved.

Now we are able to translate conditions (L) and (S) into a geometric condition concerning ranked maps.

Lemma 5. *Under the assumptions of Lemma* 4, *let, in addition,* $(\mathcal{R}_i)_{i \geq 1}$ *and* $(\mathcal{W}_i)_{i \geq 0}$ *satisfy condition* (S). *Then*:

(a) $\rho \notin \mathcal{I}(\Phi; \Sigma_{j=1}^{k} 8 \cdot 13^{k-j} e_j)$;

(b) *for any* $h > k$, $\rho \notin \mathcal{I}(\Phi; \Sigma_{j=1}^{k-1} 7 \cdot 13^{k-j} e_j + 6 e_k + e_h)$.

Proof. This is an immediate consequence of condition (S) and Lemma 4(b).

2.4. *Restatement of the results.*

Condition (S_0). Let $\mathcal{M} = (M, \mathrm{rank})$ be a ranked map. If, for every $k \geq 1$, every region Φ in M of rank k and every boundary cycle ρ of Φ, we have

(α) $\rho \notin \mathcal{I}(\Phi; \Sigma_{j=1}^{k} 8 \cdot 13^{k-j} e_j)$,

(β) for any $h > k$, $\rho \notin \mathcal{I}(\Phi; \Sigma_{j=1}^{k-1} 7 \cdot 13^{k-j} e_j + 6 e_k + e_h)$,

then we say that $\mathcal{M}$ satisfies condition (S_0).

THEOREM 3. *Let $\mathcal{M} = (M, \text{rank})$ be a connected simply-connected ranked map satisfying condition* (S_0) *and having a* reduced *boundary cycle α.*

(i) *There exist:*

(1) *a subpath β of α;*

(2) *an integer $i \geqq 1$;*

(3) *a region Φ in M, of rank i with a boundary cycle ω;*

(4) *a boundary path γ of Φ;*

(5) *simple paths $\sigma, \tau \in \mathrm{Br}(i-1)$*

such that $\beta \sim_{i-1} \sigma^{-1}\gamma\tau$ and either $\omega = \gamma\delta$ where $\delta \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 5 \cdot 13^{i-j} e_j + 4e_i)$ (see Fig. 13) *or $\gamma = \omega^m \omega'$, with $m \geqq 1$ and $\omega = \omega'\omega''$.*

(ii) *There exist:*

(1) *a subpath η of α;*

(2) *an integer $k \geqq 1$;*

(3) *a region Ψ in M of rank k;*

(4) *a boundary cycle $\eta\xi$ of Ψ*

such that either $\xi \in \mathcal{H}(\Psi; \Sigma_{j=1}^{k} 4 \cdot 13^{k-j} e_j)$ or, for some $h > k$, $\xi \in \mathcal{H}(\Psi; \Sigma_{j=1}^{k-1} 3 \cdot 13^{k-j} e_j + 2e_k + e_h)$ (see Fig. 14).

(iii) *The number of regions of M is effectively bound in terms of the length of α and the maximum of lengths of boundary cycles of regions of M.*

DEDUCTION OF THEOREM 1 FROM THEOREM 3. Let (M, L) be a minimal connected simply-connected $\mathcal{R}$-diagram for W with a boundary cycle α such

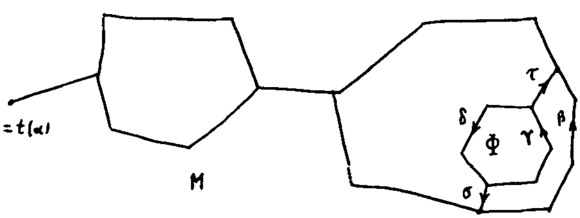


Fig. 13.

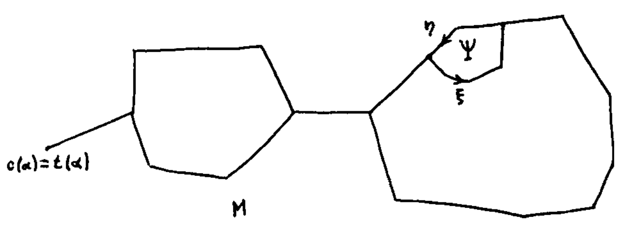


Fig. 14.

that $L(\alpha) \equiv W$ and let $\mathcal{M} = (M, \text{rank})$ be the corresponding ranked map. By Lemma 5, $\mathcal{M}$ satisfies condition (S_0). We apply Theorem 3 to $\mathcal{M}$.

(i) Take $A := L(\beta)$, $B := L(\gamma)$, $R := L(\omega)$, $Z_1 := L(\sigma)$, $Z_2 := L(\tau)$, $U := L(\delta)$, $R' := L(\omega')$, $R'' := L(\omega'')$.

Then A is a subword of $W \equiv L(\alpha)$. The relation $\beta \sim_{i-1} \sigma^{-1}\gamma\tau$ implies $A = Z_1^{-1} B Z_2 \pmod{N_{i-1}}$, hence $A^{-1} Z_1^{-1} B Z_2 \in N_{i-1}$. If $\omega = \gamma\delta$ then $R \equiv BU$ where, by Lemma 4(c),

$$U \in \mathcal{H}\left(R; \sum_{j=1}^{i-1} 5\cdot 13^{i-j} e_j + 4e_i\right).$$

Since ω is a boundary cycle of Φ, $R \equiv L(\omega) \in \mathcal{R}_i$, where $i = \text{rank}(\Phi)$. If $\gamma = \omega^m \omega'$ then $B \equiv R^m R'$ where $R \equiv R'R''$ and $m \geqq 1$.

This proves part (i).

(ii) Take $C := L(\eta)$, $S := L(\eta\xi)$, $V := L(\xi)$.

Then C is a subword of W and $S \equiv CV$. Since $\eta\xi$ is a boundary cycle of Ψ, we have $S \equiv L(\eta\xi) \in \mathcal{R}_k$ where $k = \text{rank}(\Psi)$. By Lemma 4(c), if $\xi \in \mathcal{H}(\Psi; \Sigma_{j=1}^{k} 4\cdot 13^{k-j} e_j)$, then

$$V \in \mathcal{H}\left(S; \sum_{j=1}^{k} 4\cdot 13^{k-j} e_j\right),$$

and if, for some $h > k$, $\xi \in \mathcal{H}(\Psi; \Sigma_{j=1}^{k-1} 3\cdot 13^{k-j} e_j + 2e_k + e_h)$ then

$$V \in \mathcal{H}\left(S; \sum_{j=1}^{k} 3\cdot 13^{k-j} e_j + 2e_k + e_h\right).$$

This proves part (ii).

(iii) If $\mathcal{R} = \bigcup_{i \geq 1} \mathcal{R}_i$ is finite, then the lengths of boundary cycles of regions of M do not exceed some constant l_0 depending only on $\mathcal{R}$. Then, by part (iii) of Theorem 3, the number of regions of M does not exceed some constant effectively depending on $|W| = |\alpha|$ and l_0. Therefore, up to a homeomorphism, there is only a finite number of possibilities for such an $\mathcal{R}$-diagram (M, L). Hence, given a word W, we have a finite procedure to decide whether or not $W \in N$.

This proves part (iii).

Remark. It is sufficient to prove Theorem 3 in the case when M is regular and int(M) is connected.

Indeed, given a reduced boundary cycle α of M, we can find a factorization $\alpha = \alpha_1\alpha_2\alpha_0\alpha_2^{-1}\alpha_3$ (see Fig. 15) with the following properties:

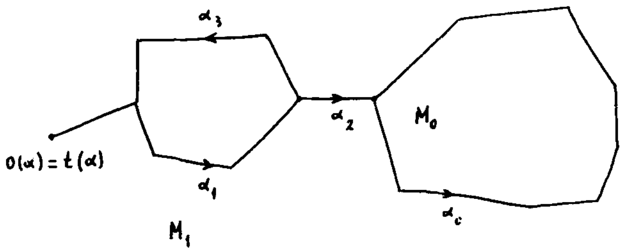


Fig. 15.

(1) α_0 is a boundary cycle of a submap M_0 of M such that M_0 is regular and $\mathrm{int}(M_0)$ is connected;

(2) $\alpha_1\alpha_3$ is a reduced boundary cycle of a submap M_1 of M;

(3) $\mathrm{Reg}(M) = \mathrm{Reg}(M_0) \cup \mathrm{Reg}(M_1)$ (cf. [1], p. 247).

Let $\mathcal{M}_i = (M_i, \mathrm{rank})$ be the ranked map such that $\mathrm{rank}_{\mathcal{M}_i}(\Phi) = \mathrm{rank}_{\mathcal{M}}(\Phi)$ for each region Φ in M_i, $i = 0, 1$.

If $\mathcal{M}$ satisfies (S_0) then, in view of Lemma 1(e), $\mathcal{M}_0$ and $\mathcal{M}_1$ also satisfy (S_0).

Using Lemma 1(e), we see that if parts (i), (ii) of Theorem 3 hold for $\mathcal{M}_0$ and α_0, they hold also for $\mathcal{M}$ and α.

In part (iii) we use induction on the length $|\alpha|$ of the boundary cycle α.

Since $|\alpha_1\alpha_3| < |\alpha|$, by the induction hypothesis, the number of regions of M_1 is effectively bounded in terms of $|\alpha_1\alpha_3|$ and l_0. If Theorem 3 holds for $\mathcal{M}_0$ then the number of regions of M_0 is effectively bounded in terms of $|\alpha_0|$ and l_0. Then the number of regions of M is effectively bounded in terms of $|\alpha|$ and l_0.

§3. Ordered 2-ranked maps and their derived maps

3.1. For technical reasons we modify the notion of a ranked map, and introduce the notion of an ordered n-ranked map.

Definition 12. Let $n \geqq 1$. An *ordered n-ranked map* is a triple $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n\}, <)$ consisting of

(1) a regular map M such that $\mathrm{int}(M)$ is connected;

(2) a partition of the set of regions of M,

$$\mathrm{Reg}(M) = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n, \qquad \mathcal{T}_i \cap \mathcal{T}_j = \varnothing \quad \text{for } i \neq j,$$

such that $\mathcal{T}_n \neq \varnothing$; and

(3) a relation of linear order "$<$" on $\mathcal{T}_2 \cup \cdots \cup \mathcal{T}_n$ such that if $\Phi \in \mathcal{T}_i$, $\Psi \in \mathcal{T}_j$ and $i < j$ then $\Phi < \Psi$ in this order.

Given a ranked map (M, rank) such that M is regular and int(M) is connected we can form an ordered n-ranked map $(M, \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}, <)$, where $n = \max\{\mathrm{rank}(\Phi) \mid \Phi \in \mathrm{Reg}(M)\}$, taking $\mathcal{T}_i := \{\Phi \mid \Phi \in \mathrm{Reg}(M), \mathrm{rank}(\Phi) = i\}$ and introducing some linear order "$<$" on the set $\mathcal{T}_2 \cup \cdots \cup \mathcal{T}_n$ such that $\Phi < \Psi$ if $\mathrm{rank}(\Phi) < \mathrm{rank}(\Psi)$.

We need some more definitions.

DEFINITION 13. *Distance between regions.* For any two regions Φ and Ψ of M contained in the same connected component of int(M), the *distance* $\mathrm{d}_M(\Phi, \Psi)$, or simply $\mathrm{d}(\Phi, \Psi)$, is defined as the minimal m such that there are regions $\Pi_0 = \Phi$, $\Pi_1, \cdots, \Pi_{m-1}$, $\Pi_m = \Psi$ and edges $e_1, \cdots, e_m$ with $e_i \subseteq \mathrm{bd}(\Pi_{i-1})$ and $e_i \subseteq \mathrm{bd}(\Pi_i)$ for $i = 1, 2, \cdots, m$. By definition, $\mathrm{d}(\Phi, \Phi) = 0$. If Φ and Ψ are contained in distinct connected components of int(M) then $\mathrm{d}(\Phi, \Psi)$ is not defined.

For example, in Fig. 16 $\mathrm{d}(\Phi_1, \Phi_3) = 2$ while $\mathrm{d}(\Psi_1, \Psi_2)$ is not defined.

The distance between regions satisfies the metric inequality

$$\mathrm{d}(\Phi, \Psi) \leqq \mathrm{d}(\Phi, \Pi) + \mathrm{d}(\Pi, \Psi).$$

If $\mathrm{d}(\Phi, \Psi) = 1$ then we call Φ and Ψ *neighbouring regions.*

DEFINITION 14. *Left-hand-side and right-hand-side factorizations of a path.* Let ν be a path in M. We say that the region Φ is *to the left* of ν if ν is a subpath of a positively oriented (in the usual sense) boundary cycle of Φ (or of its power). If ν is a subpath of a negatively oriented boundary cycle of Φ then we say that Φ is *to the right of* ν.

For example, in Fig. 17, Φ is to the left of the paths $(v_1, e_1, v_2, e_2, v_3, e_3, v_4)$ and

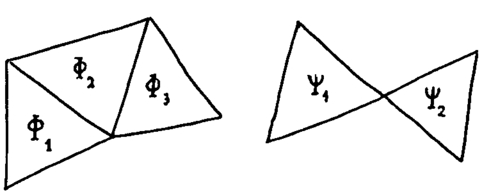


Fig. 16.

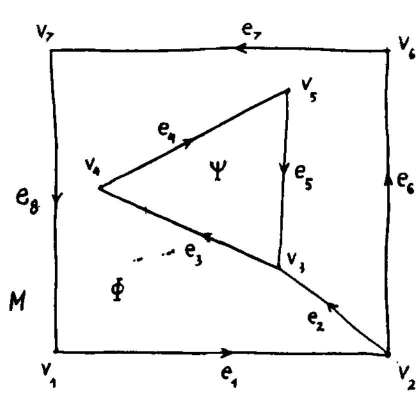


Fig. 17.

(v_2, e_2, v_3) and Φ is also to the right of (v_2, e_2, v_3), but Φ is *not* to the left of the path $(v_1, e_1, v_2, e_6, v_6)$ because this path is not a boundary path of Φ.

If clos(Φ) is simply-connected, then Φ cannot be both to the left and to the right of some boundary path of clos(Φ) (recall that all the maps considered in this paper are normalized (see Definition 5)). We may therefore say that a boundary path of Φ is positively or negatively oriented.

If Π is a connected component of the complement to M and ν a path in M, similar definitions yield the notions "Π is to the left of ν". "Π is to the right of ν", "ν is a positively (negatively) oriented boundary path of Π".

Let μ be a path in M. Traversing μ from beginning to end and checking which regions or connected components of compl(M) lie to the left of non-trivial subpaths of μ, we obtain a sequence

$$\Lambda_1(\mu), \Lambda_2(\mu), \cdots, \Lambda_m(\mu) \tag{1}$$

of regions of M or connected components of compl(M), and a factorization

$$\mu = \lambda_1(\mu)\lambda_2(\mu)\cdots\lambda_m(\mu) \tag{2}$$

such that $\Lambda_i(\mu)$ is to the left of $\lambda_i(\mu)$ and each $\lambda_i(\mu)$ is non-trivial, $i = 1, 2, \cdots, m$. For minimal m, the sequence (1) and the factorization (2) are uniquely defined. We denote this minimal m by $l(\mu)$ and we call the corresponding factorization (2) the *left-hand-side factorization* of μ in M. We stipulate that, for a trivial path $\mu = (v)$, $l(\mu) = 0$.

For example, in Fig. 17, for the path

$$\mu = (v_2, e_1^{-1}, v_1, e_1, v_2, e_6, v_6, e_7, v_7, e_8, v_1, e_1, v_2, e_2, v_3, e_5^{-1}, v_5, e_4^{-1}, v_4, e_3^{-1}, v_3, e_5^{-1}, v_5),$$

we have $l(\mu) = 4$ and

$$\Lambda_1(\mu) = \text{compl}(M), \qquad \Lambda_2(\mu) = \Phi, \qquad \Lambda_3(\mu) = \Phi, \qquad \Lambda_4(\mu) = \Psi,$$

$$\lambda_1(\mu) = (v_2, e_1^{-1}, v_1), \qquad \lambda_2(\mu) = (v_1, e_1, v_2),$$

$$\lambda_3(\mu) = (v_2, e_6, v_6, e_7, v_7, e_8, v_1, e_1, v_2, e_2, v_3),$$

$$\lambda_4(\mu) = (v_3, e_5^{-1}, v_5, e_4^{-1}, v_4, e_3^{-1}, v_3, e_5^{-1}, v_5).$$

Replacing "left" by "right" we define $r(\mu)$, the sequence

$$\mathrm{P}_1(\mu), \mathrm{P}_2(\mu), \cdots, \mathrm{P}_{r(\mu)}(\mu), \tag{3}$$

and the *right-hand-side factorization* of μ in M

$$\mu = \rho_1(\mu)\rho_2(\mu)\cdots\rho_{r(\mu)}(\mu). \tag{4}$$

3.2. *Elementary maps.*

DEFINITION 15. Let M be a regular map and Φ a fixed region in M. We call M an *elementary map over* Φ if:

(1) For each $\Psi \in \mathrm{Reg}(M)$, $\Psi \neq \Phi$, we have $d(\Phi, \Psi) = 1$.

(2) Every regular submap of M containing Φ is simply-connected.

For example, maps M_1, M_2, M_3, M_4 in Fig. 18 are elementary over Φ_1, Φ_2, Φ_3 and Φ_4 respectively, but M_1 is not elementary over Φ and M_5, M_6 are not elementary over Φ_5 and Φ_6 respectively.

LEMMA 6. *Let M be an elementary map over Φ and Ψ a region of M distinct from Φ. Then*

(a) $\mathrm{bd}(\Phi) \cap \mathrm{bd}(\Psi)$ *contains at least one edge and is connected.*

(b) $\mathrm{bd}(\Psi) \cap \mathrm{bd}(M)$ *contains at least one edge and is connected.*

(c) *There is a positively oriented boundary cycle (p.o.b.c.)* $\alpha^{-1}\gamma^{-1}\beta\delta$ *of* Ψ *such that* $\alpha = \alpha(\Psi)$ *describes* $\mathrm{bd}(\Phi) \cap \mathrm{bd}(\Psi)$ *and* $\beta = \beta(\Psi)$ *describes* $\mathrm{bd}(\Psi) \cap \mathrm{bd}(M)$ (see Fig. 19).

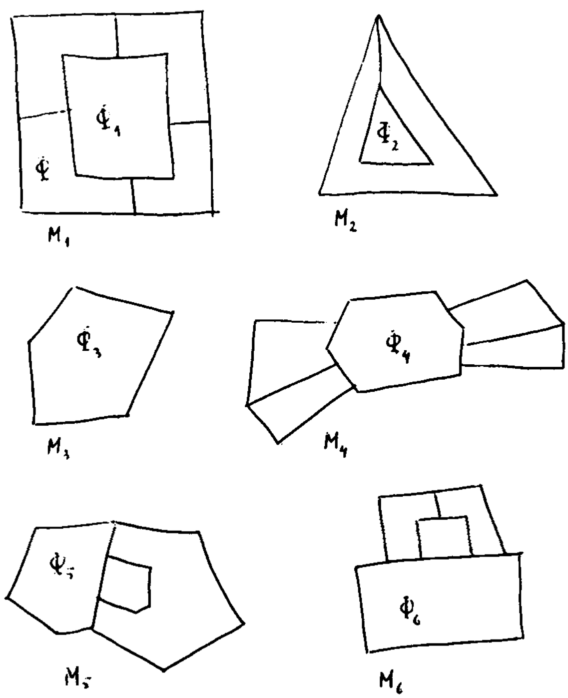


Fig. 18.

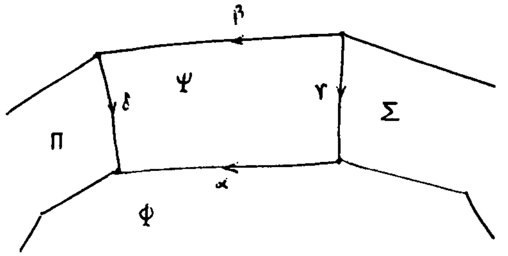


Fig. 19.

(d) *If* $\gamma = \gamma(\Psi)$ *is non-trivial then, for some region* Σ *in* M, $\Sigma \neq \Phi$, *we have* $\gamma(\Psi) = \delta(\Sigma)$.

(e) *If* $\delta = \delta(\Psi)$ *is non-trivial then, for some region* Π *in* M, $\Pi \neq \Phi$, *we have* $\delta(\Psi) = \gamma(\Pi)$.

PROOF. (a) $\mathrm{bd}(\Phi) \cap \mathrm{bd}(\Psi)$ contains at least one edge because $\mathrm{d}(\Phi, \Psi) = 1$, and it is connected because the set $\mathrm{clos}(\Phi \cup \Psi)$ is simply-connected by Definition 15.

(b) Let N be the regular submap of M containing all the regions of M except Ψ. By Definition 15, N is simply-connected; therefore $\mathrm{bd}(\Psi) \cap \mathrm{bd}(M)$ contains at least one edge, for otherwise Ψ would be contained in a bounded connected component of $\mathrm{compl}(N)$, which is impossible. Further, the complement of $\mathrm{clos}(\Psi \cup \mathrm{compl}(M))$ is connected because $\mathrm{d}(\Sigma, \Phi) = 1$ for all $\Sigma \in \mathrm{Reg}(M)$, $\Sigma \neq \Phi$. Hence $\mathrm{bd}(\Psi) \cap \mathrm{bd}(M)$ is connected.

(c) is evident, because α^{-1} and β have no edges in common.

(d) Consider the left-hand-side (l.h.s.) factorization $\gamma = \lambda_1(\gamma) \cdots \lambda_p(\gamma)$ where $p = l(\gamma)$ and let $\Lambda_1(\gamma), \cdots, \Lambda_p(\gamma)$ be the corresponding sequence. Since α describes the whole of $\mathrm{bd}(\Phi) \cap \mathrm{bd}(\Psi)$ and β describes the whole of $\mathrm{bd}(\Phi) \cap \mathrm{bd}(M)$, we have $\Lambda_i(\gamma) \neq \Phi$, $\Lambda_i(\gamma) \neq \mathrm{compl}(M)$, $i = 1, 2, \cdots, p$. We show that $p \leqq 1$.

Indeed, if $p > 1$ and $\Lambda_{p-1}(\gamma) = \Lambda_p(\gamma)$, then there is a bounded connected component (b.c.c.) of $\mathrm{compl}(\mathrm{clos}(\Lambda_p(\gamma)))$ such that $\Delta \cap \Phi = \varnothing$ (see Fig. 20). Then $\mathrm{clos}(\Phi \cup \Lambda_p(\gamma))$ is not simply-connected, which is impossible by Definition 15.

If $p > 1$ and $\Lambda_{p-1}(\gamma) \neq \Lambda_p(\gamma)$ (see Fig. 21), then $\Lambda_p(\gamma)$ is contained in the b.c.c. of $\mathrm{compl}(\mathrm{clos}(\Phi \cup \Psi \cup \Lambda_{p-1}(\gamma)))$, which also contradicts Definition 15.

Therefore, $p \leqq 1$. Since γ is non-trivial we have $p = 1$. Denote $\Lambda_1(\gamma)$ by Σ. Let us show that $\gamma = \delta(\Sigma)$.

The path γ is a boundary path of Σ satisfying the following conditions:

(1) $o(\gamma)$ is the unique vertex of γ that belongs to $\mathrm{bd}(M)$;

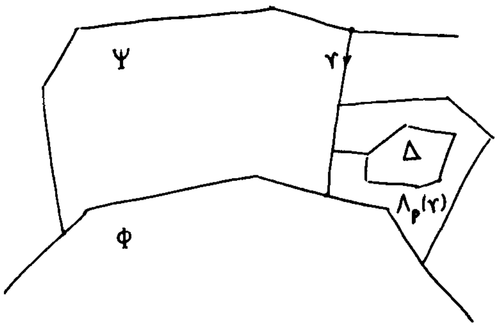


Fig. 20.

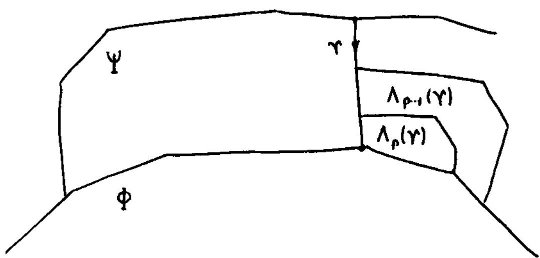


Fig. 21.

(2) $t(\gamma)$ is the unique vertex of γ that belongs to $\mathrm{bd}(\Phi)$;

(3) Σ is to the left of γ (see Fig. 22).

The path $\delta(\Sigma)$ is uniquely determined by properties (1), (2), (3), and therefore $\delta(\Sigma)=\gamma=\gamma(\Psi)$.

Part (e) of the lemma is proved in similar fashion.

The lemma is proved.

3.3. *Transversals and projections in an elementary map.*

Definition 16. *Left and right transversals from a boundary vertex.* Let M be an elementary map over a region Φ. For any vertex $v \in \mathrm{bd}(M)$ we define two paths $\mathrm{LT}(v;\Phi)$ and $\mathrm{RT}(v;\Phi)$ in the following way:

(1) If $v \in \mathrm{bd}(\Phi)$, then $\mathrm{LT}(v;\Phi):=v$, $\mathrm{RT}(v;\Phi):=v$, the trivial path (see Fig. 23).

(2) If for some region Ψ in M, $\Psi \neq \Phi$, we have $v=o(\gamma(\Psi))$, then $\mathrm{LT}(v;\Phi):=\gamma(\Psi)$, $\mathrm{RT}(v;\Phi):=\gamma(\Psi)$ (see Fig. 24).

(3) If for some region Ψ in M, $\Psi \neq \Phi$, we have $\beta(\Psi)=\mu\nu$, where μ and ν are non-trivial paths, then $\mathrm{LT}(v;\Phi):=\mu^{-1}\gamma(\Psi)$, $\mathrm{RT}(v;\Phi):=\nu\delta(\Psi)$ (see Fig. 25).

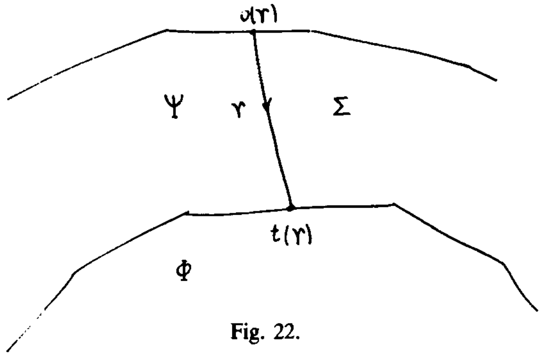


Fig. 22.

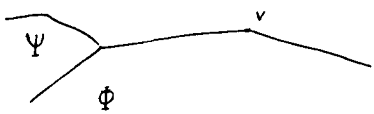


Fig. 23.

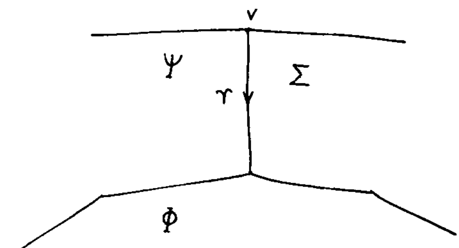


Fig. 24.

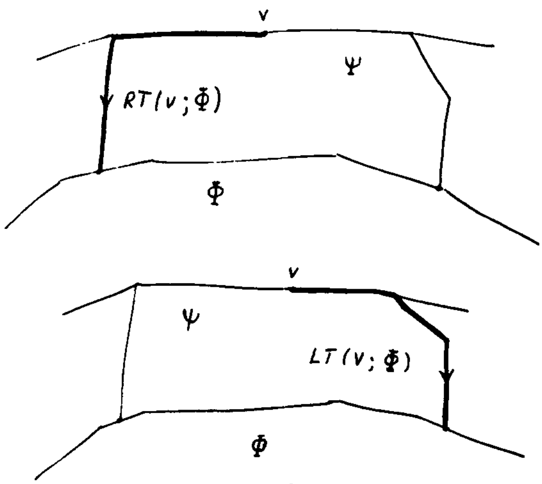


Fig. 25.

We call $\mathrm{LT}(v;\Phi)$ the *left transversal* from v to Φ and $\mathrm{RT}(v;\Phi)$ the *right transversal* from v to Φ.

Definition 17. *Left and right projections of a vertex.* Under the assumptions of the previous definition, we define two vertices $\mathrm{lpr}(v;\Phi)$ and $\mathrm{rpr}(v;\Phi)$ as follows:

$$\mathrm{lpr}(v;\Phi):=\mathrm{t}(\mathrm{LT}(v;\Phi)), \qquad \mathrm{rpr}(v;\Phi):=\mathrm{t}(\mathrm{RT}(v;\Phi)).$$

We call $\mathrm{lpr}(v;\Phi)$ the *left projection of v to Φ* and $\mathrm{rpr}(v;\Phi)$ the *right projection of v to Φ*.

Thus, $\mathrm{lpr}(v;\Phi)$ and $\mathrm{rpr}(v;\Phi)$ are distinct only in case (3) of Definition 16 (see Fig. 26), while in case (1) we have $v=\mathrm{lpr}(v;\Phi)=\mathrm{rpr}(v;\Phi)$ and in case (2) we have

$$\mathrm{lpr}(v;\Phi)=\mathrm{t}(\gamma(\Psi))=\mathrm{rpr}(v;\Phi).$$

Definition 18. *Left and right projections of a boundary path.* Let μ be a boundary path of M. Then there is a uniquely determined boundary path $\mathrm{lpr}(\mu;\Phi)$ of Φ such that

(1) $\mathrm{o}(\mathrm{lpr}(\mu;\Phi))=\mathrm{lpr}(\mathrm{o}(\mu);\Phi)$, $\mathrm{t}(\mathrm{lpr}(\mu;\Phi))=\mathrm{lpr}(\mathrm{t}(\mu);\Phi)$;

(2) $\mathrm{lpr}(\mu;\Phi)$ is homotopic to the path $\mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu\,\mathrm{LT}(\mathrm{t}(\mu);\Phi)$ in the map M_0 obtained from M by deleting the region Φ (see Fig. 27). The path $\mathrm{lpr}(\mu;\Phi)$ is called the *left projection of μ to Φ*. Replacing "left" by "right" we define the *right projection* $\mathrm{rpr}(\mu;\Phi)$ of μ to Φ.

Definition 19. *Projection of a boundary path.* Let μ be a boundary path of M. If μ is either trivial, or non-trivial and positively oriented, then we define the boundary path $\mathrm{pr}(\mu;\Phi)$ of Φ by the following two conditions:

(1) $\mathrm{o}(\mathrm{pr}(\mu;\Phi))=\mathrm{lpr}(\mathrm{o}(\mu);\Phi)$, $\mathrm{t}(\mathrm{pr}(\mu;\Phi))=\mathrm{rpr}(\mathrm{t}(\mu);\Phi)$;

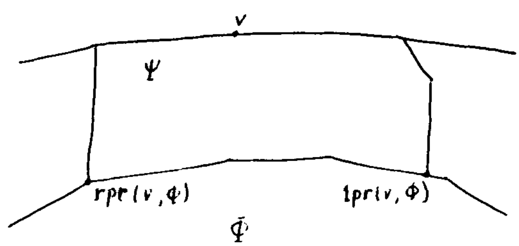


Fig. 26.

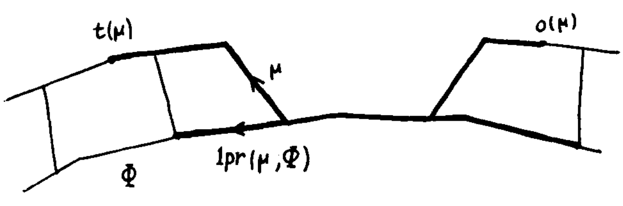


Fig. 27.

(2) $\mathrm{pr}(\mu;\Phi)$ is homotopic to the path $\mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu\,\mathrm{RT}(\mathrm{t}(\mu);\Phi)$ in the map M_0 obtained from M by deleting the region Φ (see Figs. 28 and 29).

If μ is a non-trivial negatively oriented boundary path of M then $\mathrm{pr}(\mu;\Phi) := \mathrm{pr}(\mu^{-1};\Phi)^{-1}$ (see Fig. 30). We call $\mathrm{pr}(\mu;\Phi)$ the *projection* of μ to Φ.

For example, in Fig. 31 for the path $\mu = (v_0, e_1, v_1, e_2, v_2)$ we have $\mathrm{pr}(\mu;\Phi) = (v_3, e_3, v_4, e_4, v_5, e_5, v_3, e_3, v_4, e_4, v_5, e_5, v_3)$.

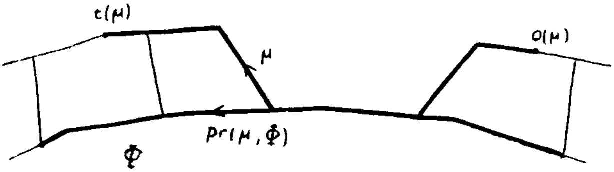


Fig. 28.

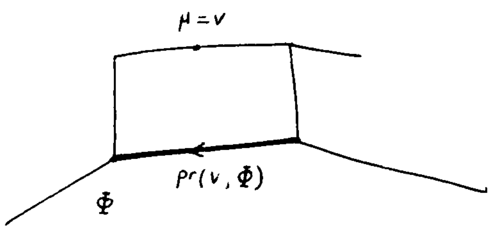


Fig. 29.

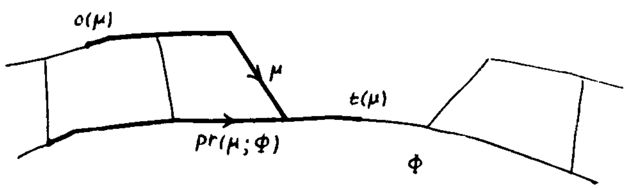


Fig. 30.

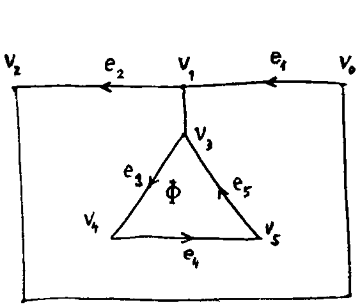


Fig. 31.

Definition 20. *Shadow of a boundary path.* Let μ be a boundary path of M. We define the *shadow of μ with respect to* Φ as the minimal submap $S(\mu;\Phi)$ of M containing the path μ and all the regions Ψ in M, $\Psi \neq \Phi$, such that $\alpha(\Psi)$ or $\alpha(\Psi)^{-1}$ is a subpath of $\mathrm{pr}(\mu;\Phi)$ (see Fig. 32) (cf. Lemma 6).

In the next lemma we collect some simple facts about projections, to be needed later on.

Lemma 7. *Let M be an elementary map over a region Φ and $\mu = \mu_1\mu_2$ a non-trivial positively oriented boundary path (p.o.b.p.) of M.*

(a) $\mathrm{pr}(\mu;\Phi)$ *is a non-trivial p.o.b.p. of* Φ.

(b) $\mathrm{lpr}(\mu;\Phi) = \mathrm{lpr}(\mu_1;\Phi)\mathrm{lpr}(\mu_2;\Phi)$.

(c) $\mathrm{rpr}(\mu;\Phi) = \mathrm{rpr}(\mu_1;\Phi)\mathrm{rpr}(\mu_2;\Phi)$.

(d) $\mathrm{pr}(\mu;\Phi) = \mathrm{lpr}(\mu_1;\Phi)\mathrm{pr}(\mu_2;\Phi) = \mathrm{pr}(\mu_1;\Phi)\mathrm{rpr}(\mu_2;\Phi)$
$= \mathrm{lpr}(\mu_1;\Phi)\mathrm{pr}(\mathrm{t}(\mu_1);\Phi)\mathrm{rpr}(\mu_2;\Phi)$.

(e) *If μ is on the boundary of Φ, then* $\mathrm{pr}(\mu;\Phi) = \mu$.

(f) *If μ is a boundary cycle of M then there are a boundary cycle ω of Φ and a boundary path τ of Φ such that* $\mathrm{pr}(\mu;\Phi) = \omega\tau = \tau\omega$.

(g) *Assume that μ is a subpath of $\beta(\Psi)$ for some region Ψ in M, $\Psi \neq \Phi$ (see Lemma* 6).
μ is a head of $\mathrm{RT}(\mathrm{o}(\mu);\Phi)$ *if and only if μ is not a head of $\beta(\Psi)$ and then* $\mathrm{RT}(\mathrm{o}(\mu);\Phi) = \mu\,\mathrm{RT}(\mathrm{t}(\mu);\Phi)$ (see Fig. 33). *Similarly, μ^{-1} is a head of* $\mathrm{LT}(\mathrm{t}(\mu);\Phi)$ *if and only if μ is not a tail of $\beta(\Psi)$ and then* $\mathrm{LT}(\mathrm{t}(\mu),\Phi) = \mu^{-1}\mathrm{LT}(\mathrm{o}(\mu);\Phi)$.

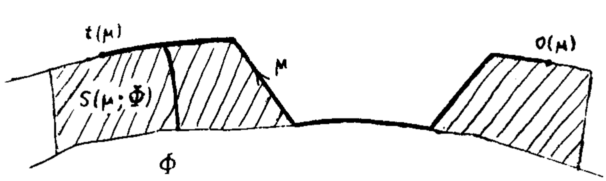


Fig. 32.

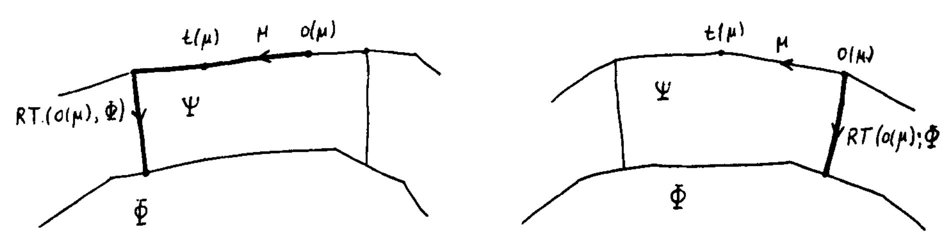


Fig. 33.

The proof of all parts of the lemma is immediate and is therefore omitted.

3.4. *Layers in an ordered 2-ranked map.*

Definition 21. Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map (see Definition 12). For any region $\Phi \in \mathcal{T}_2$ and $h \geqq 0$, we define the set of regions $\mathcal{L}^h_{\mathcal{M}}(\Phi)$, or simply $\mathcal{L}^h(\Phi)$, as follows:

$\Sigma \in \mathcal{L}^h(\Phi)$ if and only if the following holds:

(1) $\mathrm{d}(\Sigma, \Phi) = h$;

(2) for any $\Psi \in \mathcal{T}_2$, $\mathrm{d}(\Sigma, \Phi) \leqq \mathrm{d}(\Sigma, \Psi)$;

(3) if for some $\Psi \in \mathcal{T}_2$ we have $\mathrm{d}(\Sigma, \Phi) = \mathrm{d}(\Sigma, \Psi)$, then $\Phi \leqq \Psi$ in the given order relation on $\mathcal{T}_2$. (This is the only point at which the order relation on $\mathcal{T}_2$ is used.)

Let $\mathcal{L}_{\mathcal{M}}(\Phi)$, or $\mathcal{L}(\Phi)$, be the union of $\mathcal{L}^h(\Phi)$ for all $h \geqq 0$.

For example, consider the map M in Fig. 34, where we have taken $\mathcal{T}_2 = \{\Phi_1, \Phi_2, \Phi_3\}$ and $\Phi_1 < \Phi_2 < \Phi_3$. Here any region $\Phi \in \mathcal{L}^j(\Phi_i)$ is indexed by ij.

Lemma 8. (a) $\mathrm{Reg}(M) = \bigcup_{\Phi \in \mathcal{T}_2} \mathcal{L}(\Phi)$.

(b) *If* $\Phi, \Psi \in \mathcal{T}_2$ *and* $\Phi \neq \Psi$ *then* $\mathcal{L}(\Phi) \cap \mathcal{L}(\Psi) = \emptyset$.

(c) *For any* $\Phi \in \mathcal{T}_2$, $\mathcal{L}^0(\Phi) = \{\Phi\}$ *and* $\mathcal{L}^h(\Phi) \subseteq \mathcal{T}_1$, $h > 0$.

Proof. Obvious.

Lemma 9. *Let* $\Phi, \Psi \in \mathcal{T}_2$, $\Phi < \Psi$, *let* $\Gamma \in \mathcal{L}(\Phi)$, $\Delta \in \mathcal{L}(\Psi)$ *and assume that* $\mathrm{d}(\Gamma, \Delta) = 1$. *Then*

$$\mathrm{d}(\Delta, \Psi) \leqq \mathrm{d}(\Gamma, \Phi) \leqq \mathrm{d}(\Delta, \Psi) + 1.$$

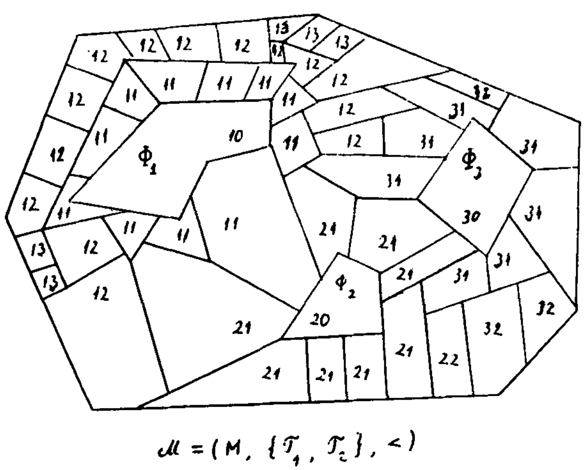


Fig. 34.

PROOF. By condition (2) of Definition 21 and the metric inequality,

$$d(\Gamma,\Phi) \leqq d(\Gamma,\Psi) \leqq d(\Gamma,\Delta) + d(\Delta,\Psi) \leqq d(\Delta,\Psi) + 1.$$

On the other hand, since $\Delta \in \mathscr{L}(\Psi)$, we have

$$d(\Delta,\Psi) \leqq d(\Delta,\Phi).$$

Since $\Phi < \Psi$, it follows from condition (3) of Definition 21 that $d(\Delta,\Psi) = d(\Delta,\Phi)$ cannot possibly be true, and so $d(\Delta,\Psi) < d(\Delta,\Phi)$. Then

$$d(\Delta,\Psi) < d(\Delta,\Psi) \leqq d(\Delta,\Gamma) + d(\Gamma,\Phi) \leqq d(\Gamma,\Phi) + 1,$$

therefore $d(\Delta,\Psi) \leqq d(\Gamma,\Phi)$.

The lemma is proved.

DEFINITION 22. For any $\Phi \in \mathscr{T}_2$ and $h \geqq 0$, let $C^h_{\mathcal{M}}(\Phi)$, or simply $C^h(\Phi)$, denote the regular submap of M such that $\mathrm{Reg}(C^h(\Phi)) = \mathscr{L}^0(\Phi) \cup \mathscr{L}^1(\Phi) \cup \cdots \cup \mathscr{L}^h(\Phi)$. Let $C_{\mathcal{M}}(\Phi)$, or $C(\Phi)$, denote the regular submap of M such that $\mathrm{Reg}(C(\Phi)) = \mathscr{L}(\Phi)$.

For example, in Fig. 35, in the situation in Fig. 34, we have indicated the submap $C^2(\Phi_1)$.

LEMMA 10. *Let* $\Phi \in \mathscr{T}_2$, $h > 0$, *and* $\Sigma \in \mathscr{L}^h(\Phi)$. *Let* $\Phi = \Pi_0, \Pi_1, \cdots, \Pi_{h-1}, \Pi_h = \Sigma$ *be regions in M such that* $d(\Pi_{i-1}, \Pi_i) = 1$, $1 \leqq i \leqq h$. *Then* $\Pi_i \in \mathscr{L}^i(\Phi)$ *for* $i = 0, 1, \cdots, h$.

PROOF. For any $i, 0 \leqq i \leqq h$, we have $d(\Pi_0, \Pi_1) \leqq i$ and $d(\Pi_i, \Pi_h) \leqq h - i$. On the other hand, by the definition of $\mathscr{L}^h(\Phi)$, $d(\Sigma,\Phi) = d(\Pi_h, \Pi_0) = h$; hence

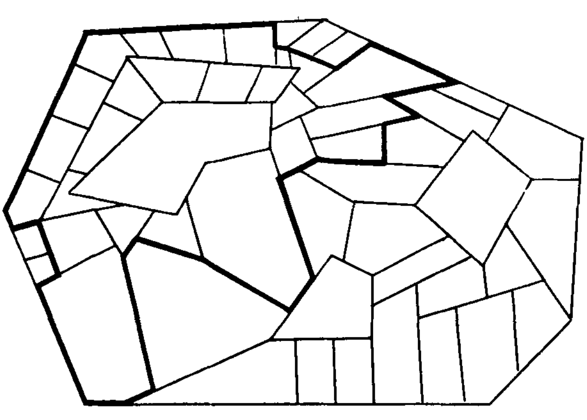


Fig. 35.

$d(\Phi, \Pi_i) = d(\Pi_0, \Pi_i) = i$ and $d(\Pi_i, \Pi_h) = d(\Pi_i, \Sigma) = h - i$ (see Fig. 36). By condition (2) of Definition 21 we have $d(\Sigma, \Psi) \geqq d(\Sigma, \Phi) = h$ for any $\Psi \in \mathcal{T}_2$. Therefore, for any $i, 0 \leqq i \leqq h$, we have

$$d(\Pi_i, \Psi) = d(\Pi_i, \Psi) + d(\Pi_i, \Sigma) - (h - i) \geqq d(\Sigma, \Psi) - (h - i)$$
$$\geqq d(\Sigma, \Phi) - (h - i) = h - (h - i) = i = d(\Pi_i, \Phi).$$

If $d(\Pi_i, \Psi) = d(\Pi_i, \Phi)$ then $d(\Sigma, \Psi) = d(\Sigma, \Phi)$ and then, by condition (3) of Definition 21, $\Phi \leqq \Psi$. Then, by Definition 21, $\Pi_i \in \mathcal{L}^i(\Phi)$. The lemma is proved.

COROLLARY. *Let* $\Phi \in \mathcal{T}_2$, $h > 0$, *and* $\Sigma \in \mathcal{L}^h(\Phi)$. *Then there is a region* $\Pi \in \mathcal{L}^{h-1}(\Phi)$ *such that* $d(\Sigma, \Pi) = 1$.

Indeed, by the definition of $\mathcal{L}^h(\Phi)$ we have $d(\Sigma, \Phi) = h$. Then there are regions $\Phi = \Pi_0, \Pi_1, \cdots, \Pi_{h-1}, \Pi_h = \Sigma$ such that $h(\Pi_{i-1}, \Pi_i) = i$, $1 \leqq i \leqq h$. By Lemma 10, $\Pi_i \in \mathcal{L}^i(\Phi)$. In particular, $\Pi_{h-1} \in \mathcal{L}^{h-1}(\Phi)$ and $d(\Sigma, \Pi_{h-1}) = d(\Pi_h, \Pi_{h-1}) = 1$. We take Π to be Π_{h-1}.

LEMMA 11. *Let* $\Phi \in \mathcal{T}_2$ *and* $h \geqq 0$. *Let* N *be a regular submap of* M *such that* $C^h(\Phi) \subseteq N \subseteq C^{h+1}(\Phi)$. *Then* int$(N)$ *is connected.*

PROOF. We shall show that each region Σ in N is contained in the same connected component of int(N) as Φ. Indeed, since $N \subseteq C^{h+1}(\Phi)$, we have $\Sigma \in \mathcal{L}^k(\Phi)$, where $k \leqq h + 1$. If $\Sigma \neq \Phi$, then $k > 0$, and we have regions $\Pi_1, \cdots, \Pi_{k-1}$ such that $d(\Pi_{i-1}, \Pi_i) = 1$, $1 \leqq i \leqq k$, where $\Pi_0 = \Phi$, $\Pi_k = \Sigma$. Then by Lemma 10, $\Pi_i \in \mathcal{L}^i(\Phi)$, hence since $k \leqq h + 1$, we have

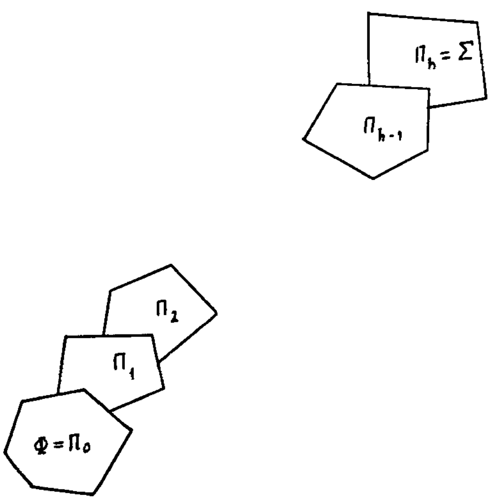


Fig. 36.

$$\Pi_0 = \Phi, \qquad \Pi_1, \cdots, \Pi_{k-1} \in \mathrm{Reg}(C^h(\Phi)) \subseteq \mathrm{Reg}(N).$$

Now the condition $d(\Pi_{i-1}, \Pi_i) = 1$, $1 \leqq i \leqq k$, implies that $\Pi_0 = \Phi$ and $\Pi_k = \Sigma$ are in the same connected component of $\mathrm{int}(N)$. The lemma is proved.

3.5. *Condition* (SC) *and the derived map of an ordered* 2*-ranked map.*

Condition (SC). Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map. We say that it satisfies condition (SC) if, for any $\Phi \in \mathcal{T}_2$ and $h \geqq 0$, every regular submap N of M such that $C^h(\Phi) \subseteq N \subseteq C^{h+1}(\Phi)$ is simply-connected.

For example, for the map M in Fig. 37, let $\mathcal{T}_2 = \{\Phi\}$. Let N be the regular submap of M with the regions Φ, Σ_1 and Σ_2. Then $C^0(\Phi) \subseteq N \subseteq C^1(\Phi)$, but N is not simply-connected. Therefore condition (SC) fails to hold. On the other hand, it is easy to see that condition (SC) is satisfied for the map in Fig. 34.

DEFINITION 23. *The regions* Φ^h *and* Φ'. Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map satisfying (SC). Then for any $\Phi \in \mathcal{T}_2$ and $h \geqq 0$, $\mathrm{int}(C^h(\Phi))$ is connected by Lemma 11. By condition (SC), $C^h(\Phi)$ is simply-connected and then also $\mathrm{int}(C^h(\Phi))$ is simply-connected, hence $\mathrm{int}(C^h(\Phi))$ is homeomorphic to the open unit square. We define the region Φ^h by

$$\Phi^h := \mathrm{int}(C^h(\Phi)). \tag{5}$$

For some $s \geqq 0$, we have $C(\Phi) = C^s(\Phi)$. Hence $\mathrm{int}(C(\Phi))$ is homeomorphic to the open unit square. We define the region Φ' by

$$\Phi' := \mathrm{int}(C(\Phi)). \tag{6}$$

For example, for the map in Fig. 34,

$$\Phi_1 = \Phi_1^0 \subseteq \Phi_1^1 \subseteq \Phi_1^2 \subseteq \Phi_1^3 = \Phi_1', \quad \Phi_2 = \Phi_2^0 \subseteq \Phi_2^1 \subseteq \Phi_2^2 = \Phi_2', \quad \Phi_3 = \Phi_3^0 \subseteq \Phi_3^1 \subseteq \Phi_3^2 = \Phi_3'.$$

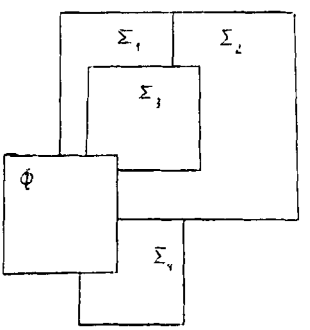


Fig. 37.

DEFINITION 24. *The derived map.* Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map satisfying condition (SC). We form a new map M' such that the regions of M' are the regions Φ' for all $\Phi \in \mathcal{T}_2$ and the vertices and edges of M' are the vertices and edges of M which lie on the boundary of some Φ'. We call M' the *derived map of* $\mathcal{M}$.

Clearly, M' is a regular map.

For example, the derived map of $\mathcal{M}$ in Fig. 34 is as shown in Fig. 38.

LEMMA 12. *M' is a normalized map.*

PROOF. Since each region Φ' of M' is of type $\Phi' = \mathrm{int}(C(\Phi))$, its boundary cannot contain vertices of degree 1, hence M' is normalized (see Definition 5). The lemma is proved.

DEFINITION 25. *The maps $E^h(\Phi)$ ($h \geqq 1$).* Let $\mathcal{M}$ be an ordered 2-ranked map satisfying condition (SC). Let $\Phi \in \mathcal{T}_2$ and $h \geqq 1$. We form a new map $E^h(\Phi)$ such that $\mathrm{Reg}(E^h(\Phi)) = \{\Phi^{h-1}\} \cup \mathcal{L}^h(\Phi)$ and the vertices and edges of $E^h(\Phi)$ are the vertices and edges of M lying on the boundary of some region of $E^h(\Phi)$.

For example, the map shown in Fig. 39 is $E^2(\Phi_1)$ for the map of Fig. 34.

LEMMA 13. *Under the conditions of Definition* 25 *$E^h(\Phi)$ is an elementary map over Φ^{h-1} (see Definition* 15).

PROOF. Let Σ be a region of $E^h(\Phi)$, $\Sigma \neq \Phi^{h-1}$. Then $\Sigma \in \mathcal{L}^h(\Phi)$. By the corollary to Lemma 10, there is a region $\Pi \in \mathcal{L}^{h-1}(\Phi)$ such that $d_M(\Sigma, \Pi) = 1$.

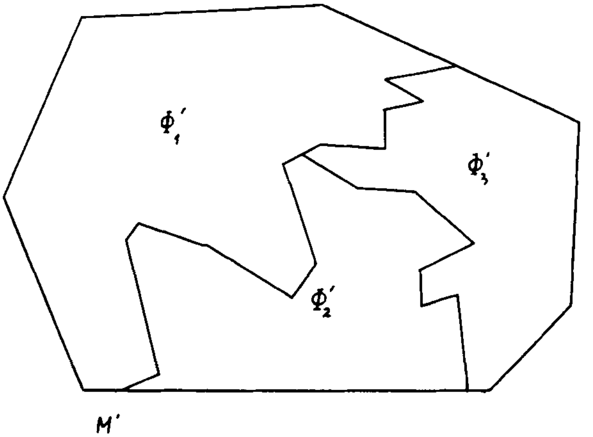


Fig. 38.

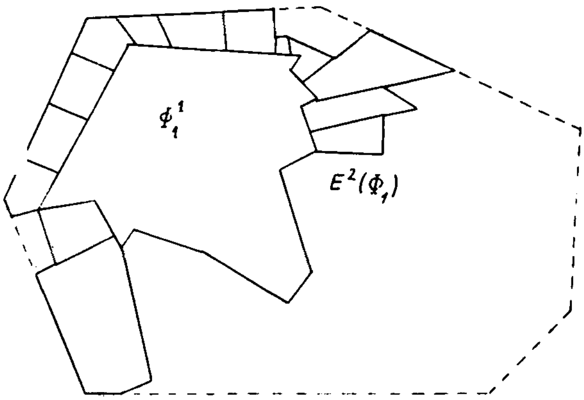


Fig. 39.

Since, by Definitions 22 and 23, $\Pi \subseteq \Phi^{h-1}$ we also have $d_{E^h(\Phi)}(\Sigma, \Phi^{h-1}) = 1$ (see Fig. 40).

Let Q be a regular submap of $E^h(\Phi)$ containing Φ^{h-1}. Deleting the regions Φ^{h-1} and adding instead the regions, edges and vertices of the interior of $C^{h-1}(\Phi)$, we obtain a map N which is a regular submap of M, satisfies $C^{h-1}(\Phi) \subseteq N \subseteq C^h(\Phi)$ and has the same support as Q. Since $\mathcal{M}$ satisfies (SC), N is simply-connected. Then Q is also simply-connected. By Definition 15, $E^h(\Phi)$ is an elementary map over Φ^{h-1}.

The lemma is proved.

Definition 26. Under the conditions of Definition 25, let $\Psi \in \mathcal{L}^h(\Phi)$. Then, considering Ψ as a region of $E^h(\Phi)$, we can define *paths* $\alpha(\Psi)$, $\beta(\Psi)$, $\gamma(\Psi)$ *and* $\delta(\Psi)$ satisfying conditions (c), (d) and (e) of Lemma 6 with M replaced by $E^h(\Phi)$ and Q replaced by Φ^{h-1}.

3.6. *Transversals and projections in an ordered 2-ranked map.* Let $\mathcal{M}$ be an ordered 2-ranked map satisfying condition (SC) and let M' be its derived map.

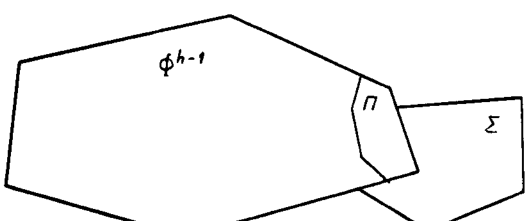


Fig. 40.

Given a boundary path μ of some region Φ' in M', we define the projection of μ to Φ as follows. We have $\mathrm{bd}(\Phi') = \mathrm{bd}(C(\Phi)) = \mathrm{bd}(C^s(\Phi)) = \mathrm{bd}(E^s(\Phi))$ for some $s \geqq 1$, and so μ is a boundary path of $E^s(\Phi)$. Since, by Lemma 13, $E^s(\Phi)$ is an elementary map over Φ^{s-1}, we can speak of the projection $\mathrm{pr}(\mu;\Phi^{s-1})$ of μ to Φ^{s-1} (see 3.3). Furthermore, $\mathrm{bd}(\Phi^{s-1}) = \mathrm{bd}(E^{s-1}(\Phi))$; hence $\mathrm{pr}(\mu;\Phi^{s-1})$ is a boundary path of $E^{s-1}(\Phi)$. We can now consider the projection of $\mathrm{pr}(\mu;\Phi^{s-1})$ to Φ^{s-2}, and so on, until we reach $\Phi^0 = \Phi$. In a similar way we can define right and left transversals and projections. The exact definitions will be given below in a more general setting.

DEFINITION 27. Let $\Phi \in \mathcal{T}_2$, $0 \leqq h \leqq l$, and let $v \in \mathrm{bd}(\Phi^l)$. The left and right projections $\mathrm{lpr}(v;\Phi^h)$ and $\mathrm{rpr}(v;\Phi^h)$ of v to Φ^h and the left and right transversals $\mathrm{LT}(v;\Phi^h)$ and $\mathrm{RT}(v;\Phi^h)$ from v to Φ^h, are defined recursively as follows:

$$\mathrm{lpr}(v;\Phi^l) := v, \quad \mathrm{lpr}(v;\Phi^{k-1}) := \mathrm{lpr}(\mathrm{lpr}(v,\Phi^k);\Phi^{k-1}), \tag{7}$$

$$\mathrm{rpr}(v;\Phi^l) := v, \quad \mathrm{rpr}(v;\Phi^{k-1}) := \mathrm{rpr}(\mathrm{rpr}(v;\Phi^k);\Phi^{k-1}), \tag{8}$$

$$\mathrm{LT}(v;\Phi^l) := v, \quad \mathrm{LT}(v;\Phi^{k-1}) := \mathrm{LT}(v;\Phi^k)\mathrm{LT}(\mathrm{lpr}(v;\Phi^k);\Phi^{k-1}), \tag{9}$$

$$\mathrm{RT}(v;\Phi^l) := v, \quad \mathrm{RT}(v;\Phi^{k-1}) := \mathrm{RT}(v;\Phi^k)\mathrm{RT}(\mathrm{rpr}(v;\Phi^k);\Phi^{k-1}) \tag{10}$$

where $1 \leqq k \leqq l$.

Let μ be a boundary path of Φ^l. We define the right and left projections $\mathrm{rpr}(\mu;\Phi^h)$ and $\mathrm{lpr}(\mu;\Phi^h)$ of μ to Φ^h and the projection $\mathrm{pr}(\mu;\Phi^h)$ of μ to Φ^h recursively, as follows:

$$\mathrm{lpr}(\mu;\Phi^l) := \mu, \quad \mathrm{lpr}(\mu;\Phi^{k-1}) := \mathrm{lpr}(\mathrm{lpr}(\mu;\Phi^k);\Phi^{k-i}), \tag{11}$$

$$\mathrm{rpr}(\mu;\Phi^l) := \mu, \quad \mathrm{rpr}(\mu;\Phi^{k-1}) := \mathrm{rpr}(\mathrm{rpr}(\mu;\Phi^k);\Phi^{k-1}), \tag{12}$$

$$\mathrm{pr}(\mu;\Phi^l) := \mu, \quad \mathrm{pr}(\mu;\Phi^{k-1}) := \mathrm{pr}(\mathrm{pr}(\mu;\Phi^k);\Phi^{k-1}) \tag{13}$$

where $1 \leqq k \leqq l$.

We define the shadow $\mathrm{S}(\mu;\Phi^h)$ of μ with respect to Φ^h recursively as follows: $\mathrm{S}(\mu;\Phi^l)$ consists of the edges and vertices of μ and

$$\mathrm{S}(\mu;\Phi^{k-1}) := \mathrm{S}(\mu;\Phi^k) \cup \mathrm{S}(\mathrm{pr}(\mu;\Phi^k);\Phi^{k-1}), \qquad 1 \leqq k \leqq l. \tag{14}$$

Since $\Phi = \Phi^0$ and $\Phi' = \Phi^s$ for some $s \geqq 0$, this definition also yields transversals, projections and shadows from Φ' to Φ.

For example, in Fig. 41 μ is a boundary path of Φ^2. We have indicated the paths $\mathrm{pr}(\mu;\Phi^1)$ and $\mathrm{pr}(\mu;\Phi)$.

LEMMA 14. *Let $k \leqq l$ and $v \in \mathrm{bd}(\Phi^l)$. Then*
(a) $\mathrm{o}(\mathrm{LT}(v;\Phi^k)) = \mathrm{o}(\mathrm{RT}(v;\Phi^k)) = v$;
(b) $\mathrm{t}(\mathrm{LT}(v;\Phi^k)) = \mathrm{lpr}(v;\Phi^k)$, $\mathrm{t}(\mathrm{RT}(v;\Phi^k)) = \mathrm{rpr}(v;\Phi^k)$.

PROOF. An immediate consequence of Definitions 17 and 27.

LEMMA 15. *Let $k \leqq l$ and let μ be a boundary path of Φ^l. Then*
(a) $\mathrm{o}(\mathrm{lpr}(\mu;\Phi^k)) = \mathrm{lpr}(\mathrm{o}(\mu);\Phi^k)$, $\mathrm{t}(\mathrm{lpr}(\mu;\Phi^k)) = \mathrm{lpr}(\mathrm{t}(\mu);\Phi^k)$;
(b) $\mathrm{o}(\mathrm{rpr}(\mu;\Phi^k)) = \mathrm{rpr}(\mathrm{o}(\mu);\Phi^k)$, $\mathrm{t}(\mathrm{rpr}(\mu;\Phi^k)) = \mathrm{rpr}(\mathrm{t}(\mu);\Phi^k)$;
(c) $\mathrm{lpr}(\mu;\Phi^k)$ *is homotopic to* $\mathrm{LT}(\mathrm{o}(\mu);\Phi^k)^{-1}\mu\,\mathrm{LT}(\mathrm{t}(\mu);\Phi^k)$ *in* $\mathrm{clos}(\Phi^l)\setminus\Phi^k$;
(d) $\mathrm{rpr}(\mu;\Phi^k)$ *is homotopic to* $\mathrm{RT}(\mathrm{o}(\mu);\Phi^k)^{-1}\mu\,\mathrm{RT}(\mathrm{t}(\mu);\Phi^k)$ *in* $\mathrm{clos}(\Phi^l)\setminus\Phi^k$.
If μ is either trivial, or non-trivial and positively oriented, then
(e) $\mathrm{o}(\mathrm{pr}(\mu;\Phi^k)) = \mathrm{lpr}(\mathrm{o}(\mu);\Phi^k)$, $\mathrm{t}(\mathrm{pr}(\mu;\Phi^k)) = \mathrm{rpt}(\mathrm{t}(\mu);\Phi^k)$;
(f) $\mathrm{pr}(\mu;\Phi^k)$ *is homotopic to* $\mathrm{LT}(\mathrm{o}(\mu);\Phi^k)^{-1}\mu\,\mathrm{RT}(\mathrm{t}(\mu);\Phi^k)$ *in* $\mathrm{clos}(\Phi^l)\setminus\Phi^k$.
If μ is non-trivial and negatively oriented, then
(g) $\mathrm{pr}(\mu;\Phi^k) = \mathrm{pr}(\mu^{-1};\Phi^k)^{-1}$;
(h) $\mathrm{o}(\mathrm{pr}(\mu;\Phi^k)) = \mathrm{rpr}(\mathrm{o}(\mu);\Phi^k)$, $\mathrm{t}(\mathrm{pr}(\mu,\Phi^k)) = \mathrm{lpr}(\mathrm{t}(\mu);\Phi^k)$;
(i) $\mathrm{pr}(\mu;\Phi^k)$ *is homotopic to* $\mathrm{RT}(\mathrm{o}(\mu);\Phi^k)^{-1}\mu\,\mathrm{LT}(\mathrm{t}(\mu);\Phi^k)$ *in* $\mathrm{clos}(\Phi^l)\setminus\Phi^k$.

PROOF. All the assertions of the lemma immediately follow from Definitions 16, 18, 19 and 27 and part (a) of Lemma 7.

LEMMA 16. *Let $\Phi \in \mathcal{T}_2$ and $k \leqq l$; let $\mu = \mu_1\mu_2$ be a non-trivial p.o.b.p. of Φ^l. Then all the assertions of Lemma 7 remain valid if Φ is replaced by Φ^k and $\Psi \in \mathcal{L}^{k+1}(\Phi) \cup \cdots \cup \mathcal{L}^l(\Phi)$.*

PROOF. An immediate consequence of Definition 27 and Lemma 7.

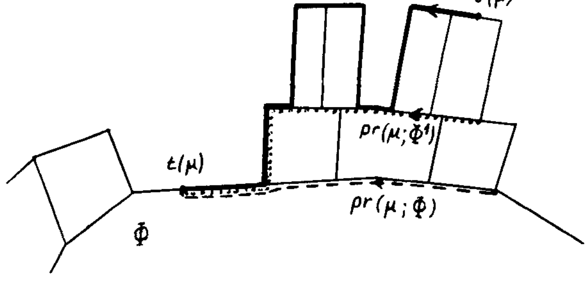


Fig. 41.

Lemma 17. *Let $\Phi \in \mathscr{T}_2$ and let μ be a p.o.b.p. of Φ^l. Let $\mu = \lambda_1(\mu)\lambda_2(\mu)\cdots\lambda_p(\mu)$ be the left-hand-side (l.h.s.) factorization of μ in M and $\Lambda_1(\mu), \Lambda_2(\mu), \cdots, \Lambda_p(\mu)$ the corresponding sequence of regions. Let*

$$l_i := d(\Lambda_i(\mu), \Phi), \qquad i = 1, 2, \cdots, P.$$

(a) *We cannot have $l_{i-1} = l_i = 0$ for some i, $1 < i \leqq P$.*
(b) *If for some $i > 1$, $l_{i-1} \leqq l_i$ then $\lambda_i(\mu)$ is a head of $\beta(\Lambda_i(\mu))$.*
(c) *If for some $i < P$, $l_{i+1} \leqq l_i$ then $\lambda_i(\mu)$ is a tail of $\beta(\Lambda_i(\mu))$.*
(d) *If for some i, $1 < i < P$, we have $l_{i-1} \leqq l_i$ and $l_{i+1} \leqq l_i$ then $\lambda_i(\mu) = \beta(\Lambda_i(\mu))$.*
(e) *μ^{-1} is a head of* LT(t(μ); Φ) *if and only if*
(α) $0 < l_1 < l_2 < \cdots < l_P$,
(β) *each $\lambda_i(\mu)$ is not a tail of $\beta(\Lambda_i(\mu))$.*
(f) *μ is a head of* RT(o(μ); Φ) *if and only if*
(α) $l_1 > l_2 > \cdots > l_P > 0$;
(β) *each $\lambda_i(\mu)$ is not a head of $\beta(\Lambda_i(\mu))$.*

Similar statements hold for a negatively oriented b.p. and its right-hand-side (r.h.s.) factorization.

Proof. (a) If $l_{i-1} = l_i = 0$ then $\Lambda_{i-1}(\mu) = \Lambda_i(\mu) = \Phi$. This can happen only if either the map M is not normalized or clos(Φ) is not simply-connected (see Fig. 42). But each of these cases is excluded, because all the maps we consider are normalized and, since $\mathcal{M}$ satisfies condition (SC) and clos(Φ) = supp($C^0(\Phi)$), clos(Φ) is simply-connected.

(b) Since $l_{i-1} \leqq l_i$, $\lambda_{i-1}(\mu)\lambda_i(\mu)$ is a p.o.b.p. of the map $E^{l_i}(\Phi)$ (see Definition 25). The region $\Lambda_i(\mu)$ belongs to $\mathscr{L}^{l_i}(\Phi)$, therefore it is a region in $E^{l_i}(\Phi)$ distinct from Φ^{l_i-1}. If $l_{i-1} = l_i$ then $\Lambda_{i-1}(\mu)$ also is a region in $E^{l_i}(\Phi)$ distinct from Φ^{l_i-1}. If $l_{i-1} < l_i$ then $\Lambda_{i-1}(\mu)$ is contained in Φ^{l_i-1} and therefore $\lambda_{i-1}(\mu)$ is a b.p. of Φ^{l_i-1} (see Fig. 43). In both cases our assertion immediately follows from Definition 26, Lemma 6 and Lemma 13.

The proof of part (c) is similar; part (d) follows from (b) and (c).

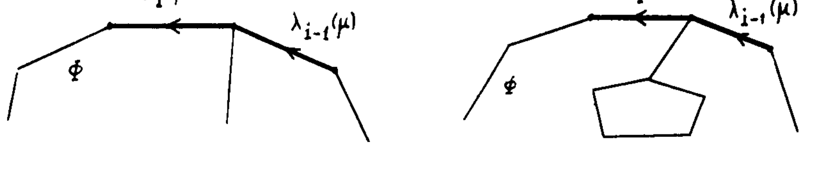


Fig. 42.

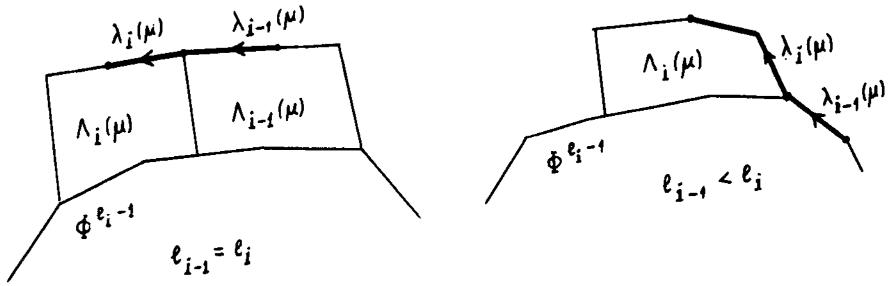


Fig. 43.

(e) Assume (α) and (β) hold. Using part (g) of Lemma 7 and Definition 27 we obtain that if $\lambda_p(\mu)^{-1}\cdots\lambda_{i+1}(\mu)^{-1}$ is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi^{l_i})$ then $\lambda_p(\mu)^{-1}\cdots\lambda_{i+1}(\mu)^{-1}\lambda_i(\mu)^{-1}$ is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi^{l_i-1})$, hence of $\mathrm{LT}(\mathrm{t}(\mu);\Phi^{l_{i-1}})$. Iterating this argument we conclude that μ^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi)$.

Reversing the above argument we obtain that if μ^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi)$ then (α) and (β) hold.

The proof of part (f) is similar.

Analogous statements for a negatively oriented b.p. and its right-hand-side factorization can be proved in similar fashion.

The lemma is proved.

Lemma 18. *Let $\Phi\in\mathcal{T}_2$, $k\leqq l$, $v\in\mathrm{bd}(\Phi^l)$; let μ be a p.o.b.p. of Φ^l such that $\mu=\mu_1\mu_2$ where μ_1^{-1} is a head of* $\mathrm{LT}(v;\Phi^k)$ *and μ_2 is a head of* $\mathrm{RT}(v;\Phi^k)$. *Then*

(a) $\mathrm{LT}(v;\Phi^k)=\mu_1^{-1}\mathrm{LT}(\mathrm{o}(\mu);\Phi^k)$;

(b) $\mathrm{RT}(v;\Phi^k)=\mu_2\mathrm{RT}(\mathrm{t}(\mu);\Phi^k)$;

(c) $\mathrm{pr}(\mu_2;\Phi^k)=\mathrm{pr}(\mu_1;\Phi^k)=\mathrm{pr}(\mu_2;\Phi^k)=\mathrm{pr}(v;\Phi^k)$. (See Fig. 44.)

Proof. An immediate consequence of Definitions 16, 27 and part (g) of Lemma 7.

Proposition 1. *Let $\mathcal{M}=(M,\{\mathcal{T}_1,\mathcal{T}_2\},<)$ be an ordered 2-ranked map satisfying condition* (SC), *M' its derived map, $\Phi\in\mathcal{T}_2$ and μ a p.o.b.p. of Φ'. Let $\mu=\lambda_1(\mu)\lambda_2(\mu)\cdots\lambda_p(\mu)$ be the l.h.s. factorization of μ in M and $\Lambda_1(\mu),\Lambda_2(\mu),\cdots,\Lambda_p(\mu)$ the corresponding sequence of regions. Let $l_i=d(\Lambda_i(\mu),\Phi)$, $1\leqq i\leqq p$.*

Assume that $\lambda_i(\mu)\neq\beta(\Lambda_i(\mu))$ for each i such that $l_i>1$. Then there is a factorization $\mu=\mu'\mu''\mu'''$ such that

(1) *μ' is a head of* $\mathrm{RT}(\mathrm{o}(\mu);\Phi)$;

(2) *$(\mu''')^{-1}$ is a head of* $\mathrm{LT}(\mathrm{t}(\mu);\Phi)$;

(3) *if μ'' is non-trivial then μ'' is on the boundary of Φ^1 and $\mu'' = \sigma_1\sigma_2\cdots\sigma_q$, where either σ_j is on the boundary of Φ or $\sigma_j = \beta(\Sigma_j)$ for some region $\Sigma \in \mathscr{L}^1(\Phi)$,* $1 \leqq j \leqq q$ (see Figs. 45 and 46).

PROOF. Write $\mu = \mu'\mu_0$, where μ' is the maximal head of μ which is also a head of $\mathrm{RT}(\mathrm{o}(\mu);\Phi)$. If μ_0^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi)$, we take $\mu'' := \mathrm{t}(\mu')$, a trivial

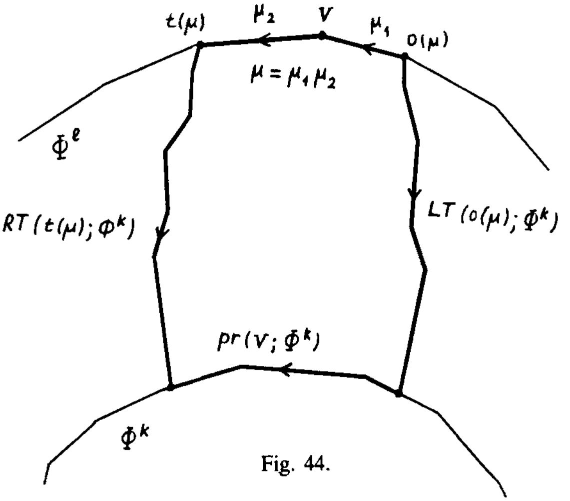


Fig. 44.

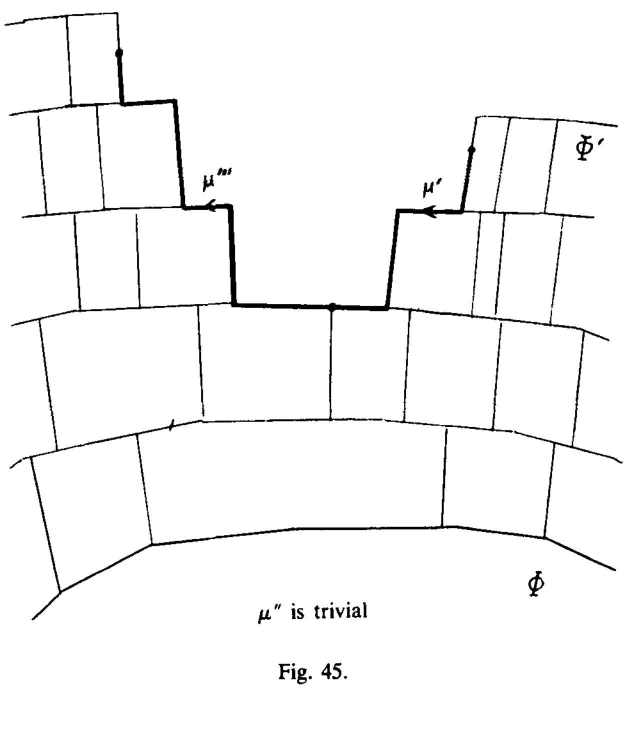
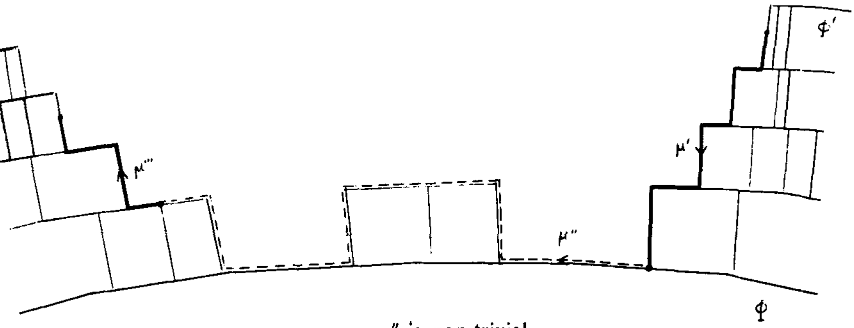


Fig. 45.



μ'' is non-trivial

Fig. 46.

path, and $\mu''' := \mu_0$ and we are done. So let us assume that μ_0^{-1} is not a head of $LT(t(\mu); \Phi)$. Then we can write $\mu_0 = \mu''\mu'''$, where μ''' is the maximal tail of μ_0 such that μ'''^{-1} is a head of $LT(t(\mu); \Phi)$ and μ'' is non-trivial.

Consider the l.h.s. factorization $\mu'' = \lambda_1(\mu'')\lambda_2(\mu'') \cdots \lambda_q(\mu'')$ of μ'' in M, and let $\Lambda_1(\mu''), \Lambda_2(\mu''), \cdots, \Lambda_q(\mu'')$ be the corresponding sequence of regions. Let $m_j := d(\Lambda_j(\mu''), \Phi)$, $1 \leqq j \leqq q$. By assumption, there is no $\Sigma \in \mathcal{L}(\Phi)$ with $d(\Sigma, \Phi) > 1$ such that $\beta(\Sigma)$ is a subpath of μ. Since μ'' is a subpath of μ, we obtain

1°. If $m_j > 1$, then $\beta(\Lambda_j(\mu'')) \neq \lambda_j(\mu'')$.

2°. If $m_1 > 0$, then $\lambda_1(\mu'')$ is a head of $\beta(\Lambda_1(\mu''))$.

Indeed, if $\lambda_1(\mu'')$ is not a head of $\beta(\Lambda_1(\mu''))$, then by part (g) of Lemma 7, $\lambda_1(\mu'')$ is a head of $RT(o(\lambda_1(\mu'')); \Phi^{m_1-1}) = RT(o(\mu''); \Phi^{m_1-1})$, hence of $RT(o(\mu''); \Phi)$. Then by part (b) of Lemma 18, $\mu'\lambda_1(\mu'')$ is a head of $RT(o(\mu); \Phi)$, contradicting the maximality of μ'.

Similarly, we have

3°. If $m_q > 0$, then $\lambda_q(\mu'')$ is a tail of $\beta(\Lambda_q(\mu''))$.

4°. Let $m = \max_j m_j$. If $m > 0$ and $m_i = m$ for some i, then, for this i, $\lambda_i(\mu'') = \beta(\Lambda_i(\mu''))$.

Indeed, if $i = 1$, then by 2°, $\lambda_i(\mu'')$ is a head of $\beta(\Lambda_i(\mu''))$. If $i > 1$, then $m_{i-1} \leqq m_i = m$, hence, by part (b) of Lemma 17, $\lambda_i(\mu'')$ is a head of $\beta(\Lambda_i(\mu''))$. Similarly, using 3° and part (c) of Lemma 17, we see that $\lambda_i(\mu'')$ is a tail of $\beta(\Lambda_i(\mu''))$. The path $\beta(\Lambda_i(\mu''))$ is either simple or a boundary cycle of Φ^m. Then the non-trivial path $\lambda_i(\mu'')$, being both a head and a tail of $\beta(\Lambda_i(\mu''))$, must coincide with it.

Comparing 1° and 4° we obtain that $m_j \leqq 1$ for $j = 1, \cdots, q$, and hence μ'' is a boundary path of Φ^1. If $m_j = 0$ then $\lambda_j(\mu'')$ is on the boundary of Φ, and if $m_j = 1$ then by 4° we have $\lambda_j(\mu'') = \beta(\Lambda_j(\mu''))$ and $\Lambda_j(\mu'') \in \mathcal{L}^1(\Phi)$. Thus, (3) is satisfied.

We have $\mu = \mu'\mu''\mu'''$, and conditions (1) and (2) are satisfied by the construction of μ', μ'' and μ'''.

This completes the proof of Proposition 1.

3.7. *Submaps.* Again, let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map satisfying condition (SC). Let N be a regular submap of M such that int(N) is connected and $\mathcal{T}_2 \cap \mathrm{Reg}(N) \neq \emptyset$. The linear order "$<$" on $\mathcal{T}_2$ induces a linear order on $\mathcal{T}_2 \cap \mathrm{Reg}(N)$, which we again denote by "$<$". Then, by Definition 12, $\mathcal{N} = (N, \{\mathcal{T}_1 \cap \mathrm{Reg}(N), \mathcal{T}_2 \cap \mathrm{Reg}(N)\}, <)$ is an ordered 2-ranked map.

The example in Fig. 47 shows that $\mathcal{N}$ need not satisfy (SC) in spite of the fact that $\mathcal{M}$ satisfies (SC). Here $\mathcal{T}_2 = \{\Phi_1, \Phi_2\}$, $\Phi_1 < \Phi_2$. In $\mathcal{M}$, $C_{\mathcal{M}}(\Phi_1)$ contains Σ_1 and Σ_3, while $C_{\mathcal{M}}(\Phi_2)$ contains Σ_2, while in $\mathcal{N}$, $C_{\mathcal{N}}(\Phi_2)$ contains Σ_1, Σ_2 and Σ_3. For the submap Q of N with the regions $\Phi_2, \Sigma_1, \Sigma_3$ we have $C^1_{\mathcal{N}}(\Phi_2) \subseteq Q \subset C^2_{\mathcal{N}}(\Phi_2)$, but Q is not simply-connected.

This example shows also that $C^h_{\mathcal{M}}(\Phi) \cap N$ may differ from $C^h_{\mathcal{N}}(\Phi)$.

We now present a sufficient condition under which the submap $\mathcal{N}$ satisfies (SC), and all the constructions of the previous section applied to $\mathcal{N}$ yield the same results as if we were working in $\mathcal{M}$. We start with the following general lemma.

Lemma 19. *Let $\Phi \in \mathcal{T}_2$ and assume that, for some $h \geqq 0$, $C^h_{\mathcal{M}}(\Phi) \subseteq N$. Then $C^{h+1}_{\mathcal{M}}(\Phi) \cap N \subseteq C^{h+1}_{\mathcal{N}}(\Phi)$.*

Proof. Let Σ be a region in $C^{h+1}_{\mathcal{M}}(\Phi) \cap N$ and let $l := d_M(\Phi, \Sigma)$. Then $l \leqq h+1$. We shall show that $\Sigma \in \mathcal{L}^l_{\mathcal{N}}(\Phi)$. There are regions $\Pi_1, \cdots, \Pi_{l-1}$ such

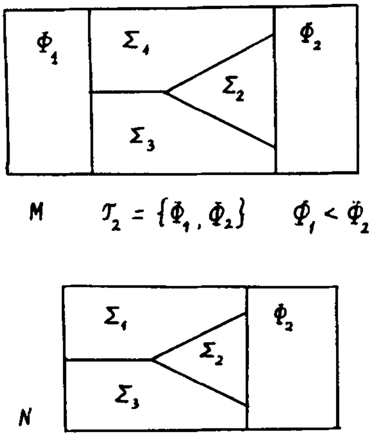


Fig. 47.

that $d_{\mathcal{M}}(\Pi_{i-1}, \Pi_i) = 1$, $1 \leqq i \leqq l$, where $\Pi_0 = \Phi$ and $\Pi_l = \Sigma$. By Lemma 10, $\Pi_i \in \mathcal{L}^i_{\mathcal{M}}(\Phi)$, $0 \leqq i \leqq l$. Since by assumption, $C^h_{\mathcal{M}}(\phi) \subseteq N$, we have $\mathcal{L}^i_{\mathcal{M}}(\Phi) \subseteq \mathrm{Reg}(N)$ for $i \leqq h$, and therefore $\Pi_i \in \mathrm{Reg}(N)$ for $i = 0, 1, \cdots, l-1$ (recall that $l \leqq h+1$). Since Π_{i-1} and Π_i are neighbouring regions in M, there is an edge on their common boundary and then they are neighbouring regions in N too. Hence $d_N(\Pi_{i-1}, \Pi_i) = 1$, $1 \leqq i \leqq l$. Thus $d_N(\Phi, \Sigma) = d_N(\Pi_0, \Pi_l) \leqq l$. On the other hand, since N is a submap of M, we have $l = d_M(\Phi, \Sigma) \leqq d_N(\Phi, \Sigma)$, therefore $d_N(\Phi, \Sigma) = l$.

Let Ψ be a region in $\mathcal{T}_2 \cap \mathrm{Reg}(N)$. Since $\Sigma \in C^{h+1}_{\mathcal{M}}(\Phi)$, we have $l = d_M(\Phi, \Sigma) \leqq d_M(\Psi, \Sigma)$ and if the equality holds then $\Phi \leqq \Psi$. But then

$$d_N(\Phi, \Sigma) = l = d_M(\Phi, \Sigma) \leqq d_M(\Phi, \Sigma) \leqq d_N(\Psi, \Sigma)$$

since N is a submap of M. If $d_N(\Phi, \Sigma) = d_N(\Psi, \Sigma)$, then also $d_M(\Phi, \Sigma) = d_M(\Psi, \Sigma)$, and therefore $\Phi \leqq \Psi$. Then, by Definition 21, $\Sigma \in \mathcal{L}^l_{\mathcal{N}}(\Phi)$, as required.

The lemma is proved.

Definition 28. Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map satisfying (SC). Let Q be a submap of M. We call Q a *1-submap* of $\mathcal{M}$ if there is a subset $\mathcal{U}$ of $\mathcal{T}_2$ such that $\mathrm{supp}(Q) = \bigcup_{\Phi \in \mathcal{U}} \mathrm{clos}(\Phi')$.

For example, putting $\mathcal{U} = \{\Phi_2, \Phi_3\}$ for the map $\mathcal{M}$ in Fig. 34, we obtain the 1-submap Q shown in Fig. 48.

Lemma 20. *If N is a regular 1-submap of $\mathcal{M}$, such that* int(N) *is connected then for each $\Phi \in \mathcal{T}_2 \cap \mathrm{Reg}(N)$ and $h \geqq 0$ we have $C^h_{\mathcal{N}}(\Phi) = C^h_{\mathcal{M}}(\Phi)$ and $C_{\mathcal{N}}(\Phi) = C_{\mathcal{M}}(\Phi)$.*

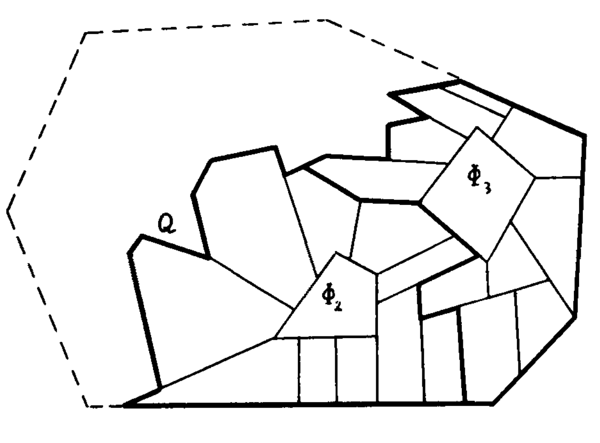


Fig. 48.

PROOF. Let $\mathscr{U}$ be a subset of $\mathscr{T}_2$ such that $\operatorname{supp}(N) = \bigcup_{\Psi\in\mathscr{U}} \operatorname{clos}(\Psi')$. Then $\mathscr{U} \subseteq \mathscr{T}_2 \cap \operatorname{Reg}(N)$. On the other hand, by the construction of the derived map M', each $\Psi' \in \operatorname{Reg}(M')$ does not contain regions from $\mathscr{T}_2$ except for Ψ, and therefore $\mathscr{U} = \mathscr{T}_2 \cap \operatorname{Reg}(N)$.

Since $\Phi \in \mathscr{T}_2 \cap \operatorname{Reg}(N) = \mathscr{U}$, we have $\operatorname{clos}(\Phi') \subseteq \operatorname{supp}(N)$. Then, for any $h \geqq 0$, $C^h_{\mathscr{M}}(\Phi) \subseteq N$. By Lemma 19, $C^{h+1}_{\mathscr{M}}(\Phi) \cap N = C^{h+1}_{\mathscr{M}}(\Phi) \subseteq C^{h+1}_{\mathscr{N}}(\Phi)$, $h \geqq 0$. Obviously, we also have $C^0_{\mathscr{M}}(\Phi) = C^0_{\mathscr{N}}(\Phi)$, the map consisting of the single region Φ and the edges and vertices on its boundary. We must now prove the inverse inclusion $C^h_{\mathscr{N}}(\Phi) \subseteq C^h_{\mathscr{M}}(\Phi)$. Let Σ be a region of $C^h_{\mathscr{N}}(\Phi)$. By Lemma 8, $\Sigma \in \mathscr{L}^k_{\mathscr{M}}(\Psi)$ for some $\Psi \in \mathscr{T}_2$ and $k \geqq 0$. In particular, $\Sigma \subseteq \Psi'$. Since $\Sigma \in \operatorname{Reg}(C^h_{\mathscr{N}}(\Phi)) \subseteq \operatorname{Reg}(N)$, it follows that $\Sigma \subseteq \operatorname{supp}(N) \cap \Psi'$. By assumption, N is a 1-submap; hence $\Psi' \subseteq \operatorname{supp}(N)$ and then $\Psi \in \operatorname{Reg}(N)$. By Definition 22, $\Sigma \in \mathscr{L}^k_{\mathscr{M}}(\Psi)$ implies that $\Sigma \in \operatorname{Reg}(C^k_{\mathscr{M}}(\Psi))$. As already shown, $C^k_{\mathscr{M}}(\Psi) \subseteq C^k_{\mathscr{N}}(\Psi)$. Comparing $\Sigma \in \operatorname{Reg}(C^h_{\mathscr{N}}(\Phi))$ and $\Sigma \in \operatorname{Reg}(C^k_{\mathscr{N}}(\Psi))$, we obtain $\Psi = \Phi$. Moreover, $k = d_M(\Sigma, \Phi) \leqq d_N(\Sigma, \Phi) \leqq h$, and so $\Sigma \in \mathscr{L}^k_{\mathscr{M}}(\Phi)$ implies $\Sigma \in \operatorname{Reg}(C^h_{\mathscr{M}}(\Phi))$. Thus, $C^h_{\mathscr{N}}(\Phi) \subseteq C^h_{\mathscr{M}}(\Phi)$. The lemma is proved.

COROLLARY. *Let N be a regular* 1-*submap of $\mathscr{M}$ such that* $\operatorname{int}(N)$ *is connected. Let* $\Phi \in \mathscr{T}_2 \cap \operatorname{Reg}(N)$. *Then*:

(a) *The ordered* 2-*ranked map* $\mathscr{N} = (N, \{\mathscr{T}_1 \cap \operatorname{Reg}(N), \mathscr{T}_2 \cap \operatorname{Reg}(N)\}, <)$ *satisfies condition* (SC) *and the derived map N' is a submap of M'.*

(b) $\mathscr{L}^h_{\mathscr{N}}(\Phi) = \mathscr{L}^h_{\mathscr{M}}(\Phi)$ $(h \geqq 0)$, $\mathscr{L}_{\mathscr{N}}(\Phi) = \mathscr{L}_{\mathscr{M}}(\Phi)$ *and* $E^h_{\mathscr{N}}(\Phi) = E^h_{\mathscr{M}}(\Phi)$ $(h \geqq 1)$.

(c) *For any* $k \leqq l$, $v \in \operatorname{bd}(\Phi^l)$ *and a b.p.* μ *of* Φ^l

$$\operatorname{lpr}_{\mathscr{N}}(v;\Phi^k) = \operatorname{lpr}_{\mathscr{M}}(v;\Phi^k), \qquad \operatorname{rpr}_{\mathscr{N}}(v;\Phi^k) = \operatorname{rpr}_{\mathscr{M}}(v;\Phi^k),$$

$$\operatorname{LT}_{\mathscr{N}}(v;\Phi^k) = \operatorname{LT}_{\mathscr{M}}(v;\Phi^k), \qquad \operatorname{RT}_{\mathscr{N}}(v;\Phi^k) = \operatorname{RT}_{\mathscr{M}}(v;\Phi^k),$$

$$\operatorname{pr}_{\mathscr{N}}(\mu;\Phi^k) = \operatorname{pr}_{\mathscr{M}}(\mu;\Phi^k), \qquad \operatorname{lpr}_{\mathscr{N}}(\mu;\Phi^k) = \operatorname{lpr}_{\mathscr{M}}(\mu;\Phi^k),$$

$$\operatorname{rpr}_{\mathscr{N}}(\mu;\Phi^k) = \operatorname{rpr}_{\mathscr{M}}(\mu;\Phi^k).$$

§4. Ordered 2-ranked maps with limitations on indices of inner regions of rank 1

4.1. DEFINITION 29. *The index of a region in a ranked map.* Let $\mathscr{M} = (M, \operatorname{rank})$ be a ranked map, let Φ be a region in M and μ a boundary path of Φ such that Φ is to the left of μ. Let

$$\mu = \rho_1(\mu)\rho_2(\mu)\cdots\rho_q(\mu) \tag{1}$$

be the r.h.s factorization of μ in M and

$$(2) \qquad P_1(\mu), P_2(\mu), \cdots, P_q(\mu)$$

the corresponding sequence of regions or connected components of compl(M). We define the *index of* Φ *in* M *relative to* μ, $\operatorname{ind}_{\mathcal{M}}(\Phi; \mu)$, or simply $\operatorname{ind}(\Phi; \mu)$, as the formal sum

$$\operatorname{ind}(\Phi; \mu) = \sum_{i \geqq 0} d_i e_i$$

where d_0 is the number of connected components of compl(M) in the sequence (2) and d_i is the number of regions of rank i in the sequence (2), each counted with its multiplicity, $i = 1, 2, \cdots$.

If Φ is to the right of μ, we define

$$\operatorname{ind}(\Phi; \mu) := \operatorname{ind}(\Phi; \mu^{-1}).$$

By the *index of a region* Φ in M, $\operatorname{ind}_{\mathcal{M}}(\Phi)$, or simply $\operatorname{ind}(\Phi)$, we mean the index of Φ relative to a positively oriented *boundary cycle* μ of Φ such that q is minimal.

It is easy to see that $\operatorname{ind}(\Phi)$ is independent of the choice of a p.o.b.c. μ of Φ with minimal q.

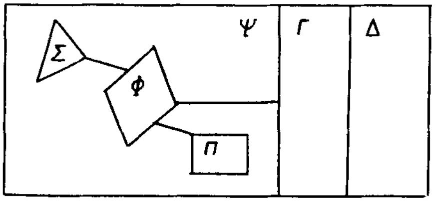
For example, let for the map $\mathcal{M}$ in Fig. 49, $\operatorname{rank}(\Pi) = \operatorname{rank}(\Psi) = 2$, and all other regions are of rank 1. Then

$$\operatorname{ind}(\Phi) = 3e_2, \quad \operatorname{ind}(\Delta) = e_0 + e_1, \quad \operatorname{ind}(\Psi) = e_0 + 6e_1 + 7e_2,$$

$$\operatorname{ind}(\Pi) = e_2, \quad \operatorname{ind}(\Sigma) = e_2, \quad \operatorname{ind}(\Gamma) = 2e_0 + e_1 + 2e_2.$$

Let $d = \sum_{i \geqq 0} d_i e_i$, $f = \sum_{i \geqq 0} f_i e_i$. We write $d \leqq f$ if $d_i \leqq f_i$ for $i \geqq 0$.

Definition 30. Let $\Phi \in \operatorname{Reg}(M)$, $d = \sum_{i \geqq 0} d_i e_i$. Let μ be a boundary path of Φ. If $\operatorname{ind}(\Phi; \mu) \leqq d$, we write $\mu \in \Phi(d)$.



$\mathcal{M} = (M, \operatorname{rank})$

Fig. 49.

Comparing Definition 9 and Definition 30, we obtain

LEMMA 21. *Let* Φ *be a region in* M *such that* $\mathrm{clos}(\Phi)$ *is simply-connected. Let* μ *be a b.p. of* Φ.

(a) *If* $\mathrm{rank}(\Phi)=1$, *then for any* $d=\Sigma_{i\geqq 1} d_i e_i$, $\mu\in I(\Phi;d)$ *if and only if* $\mu\in\Phi(d)$.

(b) *For any* $m\geqq 0$, $\mu\in I(\Phi, me_1)$ *if and only if* $\mu\in\Phi(me_1)$.

The proof is obvious.

DEFINITION 31. *Inner region of* M. We call a region Φ in M an *inner region* if $\mathrm{bd}(\Phi)\cap\mathrm{bd}(M)$ contains no edges.

Let $\mathrm{ind}(\Phi)=\Sigma_{i\geqq 0} d_i e_i$. Φ is an inner region if and only if $d_0=0$.
Thus, for example, in Fig. 50 Φ is an inner region in M.

Conditions $D(p)$ *and* $D(q;1)$. Let $\mathcal{M}=(M,\mathrm{rank})$ be a ranked map. We say that $\mathcal{M}$ satisfies condition $D(p)$ if it contains no region Φ of rank 1 such that $\mathrm{ind}(\Phi)\leqq pe_1$. We say that $\mathcal{M}$ satisfies $D(q;1)$ if it contains no region Φ of rank 1 such that $\mathrm{ind}(\Phi)\leqq qe_1+e_2$.

4.2. Let $\mathcal{M}=(M,\{\mathcal{T}_1,\mathcal{T}_2\},<)$ be an ordered 2-ranked map satisfying condition (SC).

LEMMA 22. *Let* $\Phi\in\mathcal{T}_2$, $h\geqq 1$ *and* $\Pi\in\mathcal{L}^h(\Phi)$. *Then*:

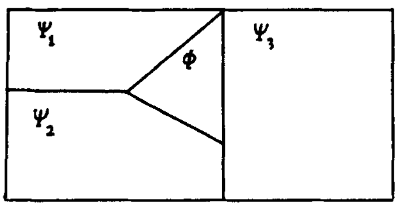
(a) $\gamma(\Pi)\in\Pi(e_1)$ *and* $\delta(\Pi)\in\Pi(e_1)$ (see Definition 26).

(b) *If* $h=1$, *then* $\alpha(\Pi)\in\Pi(e_2)$ *and*

$$\mathrm{ind}(\Pi)\leqq 2e_1+e_2+\mathrm{ind}(\Pi;\beta(\Pi)).$$

(c) *If* $\mathcal{M}$ *satisfies* $D(5)$ *and* $D(3;1)$, *then for each* $h>1$, $\alpha(\Pi)\in\Pi(2e_1)$ *and*

$$\mathrm{ind}(\Pi)\leqq 4e_1+\mathrm{ind}(\Pi;\beta(\Pi)).$$



M

Fig. 50.

PROOF. (a) and the first statement of (b) follow immediately from Definition 26 and Lemma 6. The second statement of (b) follows from the obvious inequality

(3) $\operatorname{ind}(\Pi) \leqq \operatorname{ind}(\Pi; \alpha(\Pi)) + \operatorname{ind}(\Pi; \beta(\Pi)) + \operatorname{ind}(\Pi; \gamma(\Pi)) + \operatorname{ind}(\Pi; \delta(\Pi))$.

(c) We use induction on h. Let $\alpha(\Pi) = \lambda_1\lambda_2 \cdots \lambda_p$ be the l.h.s. factorization of $\alpha(\Pi)$ in M and $\Lambda_1, \Lambda_2, \cdots, \Lambda_p$ the corresponding sequence of regions, where $p = l(\alpha(\Pi))$, $\lambda_i = \lambda_i(\alpha(\Pi))$, $\Lambda_i = \Lambda_i(\alpha(\Pi))$. Since $\Pi \in \mathscr{L}^h(\Phi)$, we have $\Lambda_i \in \mathscr{L}^{h-1}(\Phi)$, $1 \leqq i \leqq p$. By Lemma 8(c), $\Lambda_i \in \mathscr{T}_1$ and then $\operatorname{ind}(\Pi, \alpha(\Pi)) = pe_1$. (See Fig. 51.)

If $p > 2$, then $\lambda_2 = \beta(\Lambda_2)$ and so $\operatorname{ind}(\Lambda_2, \beta(\Lambda_2)) = e_1$. If $h = 2$, then, applying part (b) to Λ_2, we obtain $\operatorname{ind}(\Lambda_2) \leqq 3e_1 + e_2$, contradicting D(3; 1).

If $h > 2$, then, applying the induction hypothesis to Λ_2, we obtain $\operatorname{ind}(\Lambda_2) \leqq 5e_1$, contradicting D(5). Therefore $p \leqq 2$ and so $\operatorname{ind}(\Pi; \alpha(\Pi)) = pe_1 \leqq 2e_1$. The second statement follows from (3).

The lemma is proved.

LEMMA 23. *Assume that M satisfies* D(6) *and* D(4; 1). *Let $\Phi \in \mathscr{T}_2$ and let μ be a p.o.b.p. of Φ^{h-1}, $h > 1$. If μ is a subpath of $\alpha(\Pi_1)\alpha(\Pi_2)$ for some $\Pi_1, \Pi_2 \in \mathscr{L}^h(\Phi)$, then for the l.h.s. factorization*

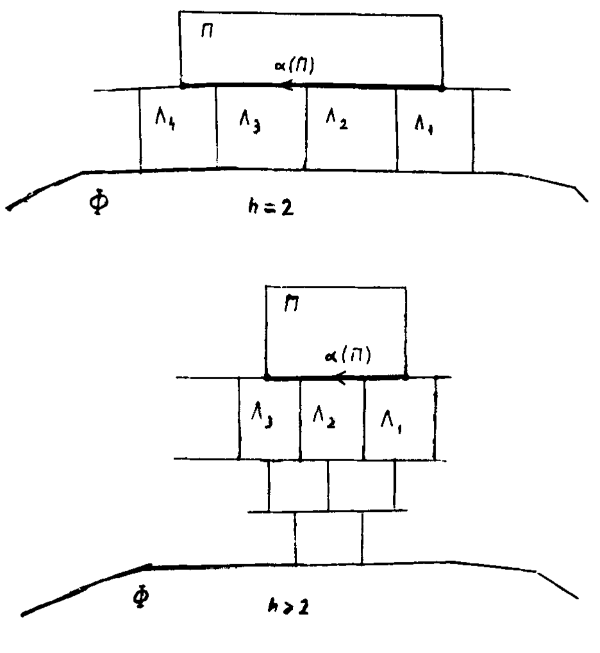


Fig. 51.

(4) $$\mu = \lambda_1\lambda_2\cdots\lambda_p$$

of μ *in* M *we have* $p \leqq 2$.

PROOF. We argue as in the proof of Lemma 22. Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_p$ be the sequence of regions corresponding to the factorization (4). If $p > 2$ then $\lambda_2 = \beta(\Lambda_2)$ and then for $h = 2$, $\operatorname{ind}(\Lambda_2) \leqq 4e_1 + e_2$, contradicting D(4; 1), and for $h > 2$, $\operatorname{ind}(\Lambda_2) \leqq 6e_1$, contradicting D(6) (see Lemma 22 (b) and (c)). Therefore, $p \leqq 2$, as required.

LEMMA 24. *Assume that* $\mathcal{M}$ *satisfies* D(6) *and* D(4; 1). *Let* $\Phi \in \mathcal{T}_2$. *If* μ *is a subpath of* $\beta(\Pi)$ *for some* $\Pi \in \mathcal{L}^h(\Phi)$, $h \geqq 1$, *then*:

(a) *either* $\operatorname{lpr}(\mu;\Phi)$ *is trivial or* $\operatorname{lpr}(\mu;\Phi) = \alpha(\Sigma_1)$ *for some* $\Sigma_1 \in \mathcal{L}^1(\Phi)$;

(b) *either* $\operatorname{rpr}(\mu;\Phi)$ *is trivial or* $\operatorname{rpr}(\mu;\Phi) = \alpha(\Sigma_2)$ *for some* $\Sigma_2 \in \mathcal{L}^1(\Phi)$;

(c) *either* $\operatorname{pr}(\mu;\Phi)$ *is trivial or* $\operatorname{pr}(\mu;\Phi) = \alpha(\Sigma_3)$ *or else* $\operatorname{pr}(\mu;\Phi) = \alpha(\Sigma_3)\alpha(\Sigma_4)$ *for some* $\Sigma_3, \Sigma_4 \in \mathcal{L}^1(\Phi)$.

In particular, for any vertex $v \in \operatorname{bd}(\Phi')$, $\operatorname{lpr}(v;\Phi) \in \Phi(e_1)$, $\operatorname{rpr}(v;\Phi) \in \Phi(e_1)$ *and* $\operatorname{pr}(v;\Phi) \in \Phi(2e_1)$.

PROOF. For some $k \leqq h$, let us consider $\operatorname{pr}(\mu;\Phi^k)$. By induction on $h - k$ we show that either $\operatorname{pr}(\mu;\Phi^k)$ is trivial or $\operatorname{pr}(\mu;\Phi^k) = \alpha(\Gamma_1)$ or else $\operatorname{pr}(\mu;\Phi^k) = \alpha(\Gamma_1)\alpha(\Gamma_2)$ for some $\Gamma_1, \Gamma_2 \in \mathcal{L}^{k+1}(\Phi)$.

Indeed, for $k = h$, $\operatorname{pr}(\mu;\Phi^h) = o(\alpha(\Pi))$ for $\mu = o(\beta(\Pi))$, $\operatorname{pr}(\mu;\Phi^h) = t(\alpha(\Pi))$ for $\mu = t(\beta(\Pi))$ and $\operatorname{pr}(\mu;\Phi^k) = \alpha(\Pi)$ otherwise.

If $\operatorname{pr}(\mu;\Phi^k) = \alpha(\Gamma_1)\alpha(\Gamma_2)$ for some $\Gamma_1, \Gamma_2 \in \mathcal{L}^{k+1}(\Phi)$ and $k > 0$ then according to Lemma 23, the path $\alpha(\Gamma_1)\alpha(\Gamma_2)$ is a subpath either of $\beta(\Delta_1)$ or of $\beta(\Delta_1)\beta(\Delta_2)$ (but not of $\beta(\Delta_1)$ or $\beta(\Delta_2)$) for some $\Delta_1, \Delta_2 \in \mathcal{L}^k(\Phi)$ (see Fig. 52).

In the first case $\operatorname{pr}(\alpha(\Gamma_1)\alpha(\Gamma_2);\Phi^{k-1}) = \alpha(\Delta_1)$ and in the second case $\operatorname{pr}(\alpha(\Gamma_1)\alpha(\Gamma_2);\Phi^{k-1}) = \alpha(\Delta_1)\alpha(\Delta_2)$.

If $\operatorname{pr}(\mu;\Phi^k) = \alpha(\Gamma_1)$ and $k > 0$ then a similar argument applies (see Fig. 53).

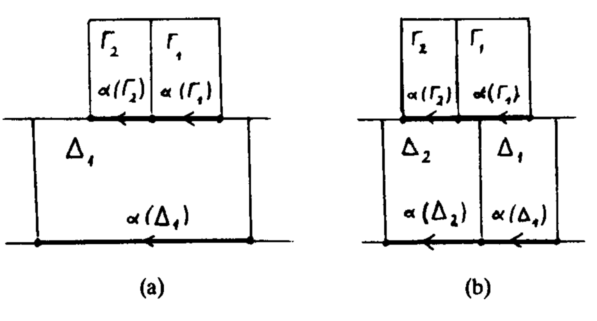
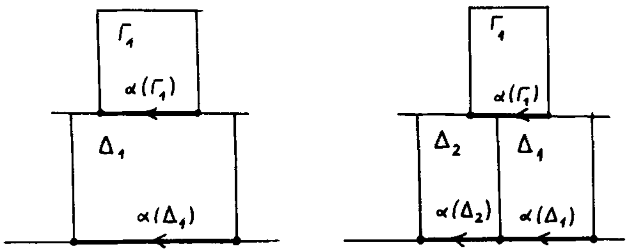


Fig. 52.



(a) (b)

Fig. 53.

If $\mathrm{pr}(\mu;\Phi^k)$ is trivial and $k>0$ then either $\mathrm{pr}(\mu;\Phi^{k-1})$ is trivial or $\mathrm{pr}(\mu;\Phi^{k-1})=\alpha(\Lambda)$ for some $\Delta\in\mathscr{L}^k(\Phi)$. This proves part (c).

Parts (a) and (b) can be proved in a similar way. We have only to observe that in Fig. 53(b)

$$\mathrm{lpr}(\alpha(\Gamma_1);\Phi^k)=\alpha(\Delta_1),\qquad \mathrm{rpr}(\alpha(\Gamma_1);\Phi^k)=\alpha(\Delta_2).$$

The lemma is proved.

4.3. *Paths on the common boundary of two regions in the derived map.* Let $\mathcal{M}=(M,\{\mathcal{T}_1,\mathcal{T}_2\},<)$ be an ordered 2-ranked map satisfying conditions (SC), D(8) and D(6; 1).

LEMMA 25. *Let $\Phi,\Psi\in\mathcal{T}_2$; let $\Gamma\in\mathscr{L}(\Phi)$ and let τ be a boundary path of Γ and also a boundary path of Ψ'. Let $l=\mathrm{d}(\Gamma,\Phi)$. Assume that one of the following conditions holds:*

(α) *$\Psi<\Phi$ and $l\geqq 1$ (i.e. $\Gamma\neq\Phi$),*

(β) *$\Phi<\Psi$ and $l>1$,*

(γ) *$\Phi<\Psi$, $l=1$ and τ does not contain boundary edges of Ψ.*

Then $\mathrm{ind}(\Gamma,\tau)\leqq 4e_1$.

REMARK. For this lemma we actually need only (SC), D(5) and D(3; 1).

PROOF. Without loss of generality, we may assume that τ is non-trivial and Ψ' is to the left of τ. Let $\tau=\lambda_1\lambda_2\cdots\lambda_p$ be the l.h.s. factorization of τ in M and let $\Lambda_1,\Lambda_2,\cdots,\Lambda_p$ be the corresponding sequence of regions, where $p=l(\tau)$, $\lambda_i=\lambda_i(\tau)$, $\Lambda_i=\Lambda_i(\tau)$, $1\leqq i\leqq p$. We denote $l_i=\mathrm{d}(\Lambda_i,\Psi)$, $1\leqq i\leqq p$ (see Fig. 54).

1°. $l_i\geqq 1$ for all i, $1\leqq i\leqq p$.

Indeed, in case (α) we have $l_i\geqq l\geqq 1$ by Lemma 9. In case (β), an application of Lemma 9 gives $l_i\geqq l-1\geqq 1$. In case (γ), $\Lambda_i\neq\Psi$ for all l, because otherwise τ would contain a boundary edge of Ψ. Therefore $l_i=\mathrm{d}(\Lambda_i,\Psi)\geqq 1$.

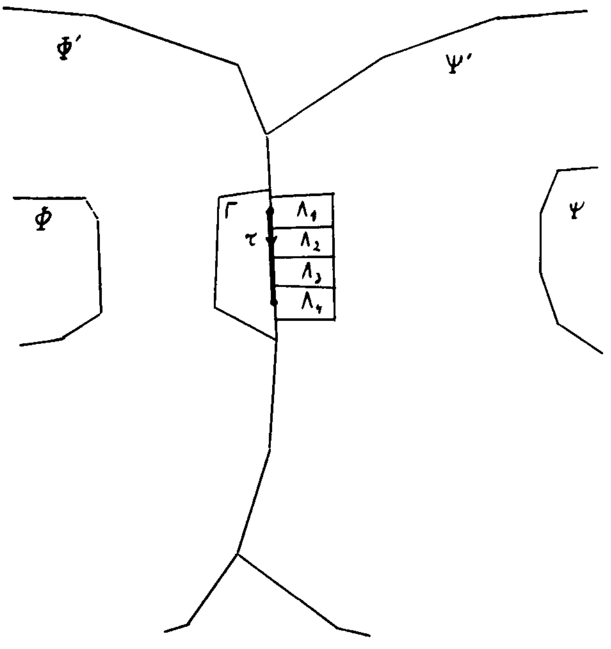


Fig. 54.

In particular, using Lemma 8(c), we obtain $\Lambda_i \in \mathcal{T}_1$ for all i. Therefore:

2°. $\operatorname{ind}(\Gamma; \tau) = pe_1$.

3°. There exists m such that $l_i = m - 1$ or m for all i.

Indeed, in case (α), $l \leqq l_i \leqq l + 1$ by Lemma 3 and we take $m = l + 1$. In cases (β) and (γ) the same lemma gives $l_i \leqq l \leqq l_i + 1$; hence $l - 1 \leqq l_i \leqq l$ and we take $m = l$.

Because of 2° we have to show that $p \leqq 4$. Suppose that $p > 4$, and consider the sequence of numbers $l_1, l_2, \cdots, l_p$. We say that for some $j, 1 < j < p$, l_j is a *weak local maximum* if $l_{j-1} \leqq l_i$ and $l_{j+1} \leqq l_j$. It is easy to verify that the longest sequence taking only two values and having no weak local maxima is $m, m-1, m-1, m$. Hence there exists $j, 1 < j < p$, such that $l_{j-1} \leqq l_j$ and $l_{j+1} \leqq l_j$. Then, by Lemma 17(d), $\beta(\Lambda_j) = \lambda_j$. Since $l \geqq 1$, we have $\Gamma \in \mathcal{T}_1$, hence $\operatorname{ind}(\Lambda_j, \beta(\Lambda_j)) = e_1$.

If $l_j = 1$ then, by Lemma 22(b), $\operatorname{ind}(\Lambda_j) \leqq 3e_1 + e_2$, contradicting D(6; 1), and if $l_j > 1$ then, by Lemma 22(c), $\operatorname{ind}(\Lambda_j) \leqq 5e_1$, contradicting D(8).

Therefore necessarily $p \leqq 4$. Thus $\operatorname{ind}(\Gamma, \tau) = pe_1 \leqq 4e_1$.

The lemma is proved.

PROPOSITION 2. *Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map satisfying conditions* (SC), D(8) *and* D(6; 1). *Let $\Phi, \Psi \in \mathcal{T}_2$. Let μ be a non-trivial p.o.b.p. of Φ' which is simultaneously a n.o.b.p. of Ψ'. Then there is a factorization*

$$\mu = \mu'\mu''\mu''' \tag{5}$$

and, if μ'' is non-trivial, a further factorization

$$\mu'' = \mu_1\mu_2\cdots\mu_h \tag{6}$$

such that

(a) *μ' is a head of* $\mathrm{RT}(\mathrm{o}(\mu); \Phi)$.

(b) *μ'''^{-1} is a head of* $\mathrm{LT}(\mathrm{t}(\mu); \Phi)$.

(c) $\mathrm{pr}(\mu'; \Phi) \in \Phi(2e_1)$, $\mathrm{pr}(\mu'''; \Phi) \in \Phi(2e_1)$.

(d) $\mathrm{pr}(\mu'; \Psi) \in \Psi(4e_1)$, $\mathrm{pr}(\mu'''; \Psi) \in \Psi(4e_1)$.

If $\Phi < \Psi$ in the order on $\mathcal{T}_2$ and μ'' is non-trivial then

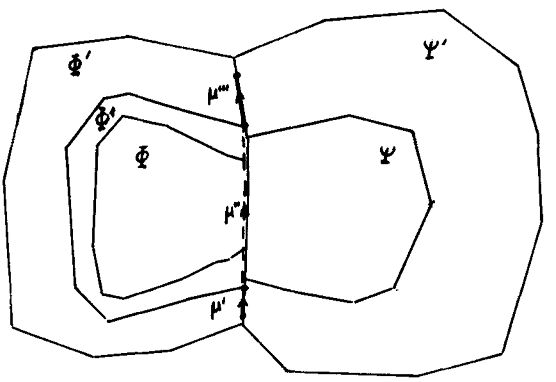
(e) *μ'' is on the boundary of Φ^1* (see Definition 23).

(f) *Each μ_j contains a boundary edge of Ψ, and if $h \geqq 2$ then $\mu_2 \cdots \mu_{h-1}$ is on the boundary of Ψ.*

(g) *The factorization* (6) *is the l.h.s. factorization of μ'' in M.*

(h) *For each j, $1 \leqq j \leqq h$, either μ_j is on the common boundary of Φ and Ψ or $\mu_j = \beta(\Gamma_j)$ for some $\Gamma_j \in \mathcal{L}^1(\Phi)$.*

(i) *If $\mu_1 = \beta(\Gamma_1)$ then $\mathrm{pr}(\mu'\mu_1; \Psi) \in \Psi(5e_1)$ and if $\mu_h = \beta(\Gamma_h)$ then* $\mathrm{pr}(\mu_h\mu'''; \Psi) \in \Psi(5e_1)$ (see Fig. 55).



$\Phi < \Psi$

Fig. 55.

If $\Psi < \Phi$ and μ'' is non-trivial, then instead of (e), (f), (g), (h), (i) *we have the following*:

(e′) *μ'' is on the boundary of Φ.*

(f′) *μ'' is on the boundary of Ψ^1.*

(g′) *The factorization* (6) *is the r.h.s. factorization of μ'' in M.*

(h′) *For each j, $1 < j < h$, either μ_j is on the common boundary of Φ and Ψ or $\mu_j = \beta(\Pi_j)^{-1}$ for some $\Pi_j \in \mathcal{L}^1(\Psi)$. If μ_1 is not on the common boundary of Φ and Ψ, then μ_1 is a subpath of $\beta(\Pi_1)^{-1}$ for some $\Pi_1 \in \mathcal{L}^1(\Psi)$. If μ_h is not on the common boundary of Φ and Ψ, then μ_h is a subpath of $\beta(\Pi_2)^{-1}$ for some $\Pi_h \in \mathcal{L}^1(\Psi)$.*

(i′) *If μ_1 is not on the boundary of Ψ, then* $\mathrm{pr}(\mu'\mu_1;\Psi) \in \Psi(5e_1)$; *if μ_h is not on the boundary of Ψ, then* $\mathrm{pr}(\mu_h\mu''';\Psi) \in \Psi(5e_i)$ (see Fig. 56).

PROOF. Since $\mathcal{M}$ satisfies (SC), clos(Φ') is simply-connected and therefore the fact that μ is a non-trivial p.o.b.p. of Φ' and a n.o.b.p. of Ψ' implies that $\Phi' \neq \Psi'$; hence $\Phi \neq \Psi$.

We have:

1°. Let $\Gamma \in \mathcal{L}(\Phi)$ and let τ be a boundary path of Γ which is a subpath of μ. Suppose that one of conditions (α), (β), (γ) of Lemma 25 holds. Then $\tau \neq \beta(\Gamma)$.

Indeed, by Lemma 25, $\tau \in \Gamma(4e_1)$. If $\tau = \beta(\Gamma)$ then, by Lemma 22, if $l = \mathrm{d}(\Gamma, \Phi) = 1$ then $\mathrm{ind}(\Gamma) \leqq 6e_1 + e_2$ contradicting D(6; 1) and if $l = \mathrm{d}(\Gamma, \Phi) > 1$ then $\mathrm{ind}(\gamma) \leqq 8e_1$ contradicting D(8). Therefore $\tau \neq \beta(\Gamma)$, as required.

We are now in a position to apply Proposition 1 to the regions Φ, Φ' and the path μ. We obtain factorization (5) with the following properties:

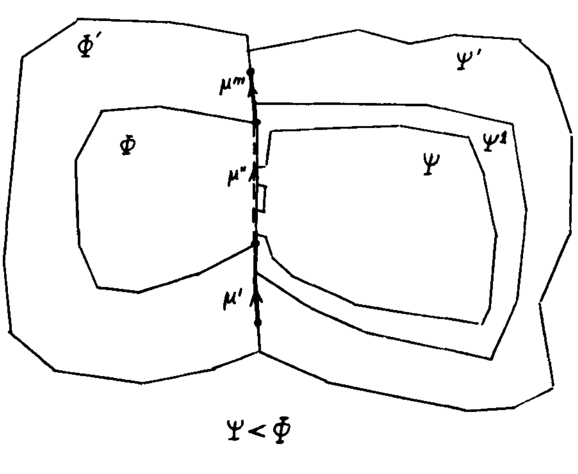


Fig. 56.

2°. μ' is a head of $\mathrm{RT}(\mathrm{o}(\mu);\Phi)$.

3°. μ'''^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi)$.

4°. If μ'' is non-trivial, then μ'' is on the boundary of Φ^1 and there is a factorization

$$\mu'' = \sigma_1\sigma_2\cdots\sigma_q \tag{7}$$

such that for each j, $1 \leqq j \leqq q$, either σ_j is on the boundary of Φ or $\sigma_j = \beta(\Gamma_j)$ for some $\Gamma_j \in \mathscr{L}^1(\Phi)$. Moreover, we may assume without loss of generality that (7) is the l.h.s. factorization of μ'' in M.

Comparing 1° with 4°, we reach the following conclusions:

5°. If $\Psi < \Phi$ and μ'' is non-trivial then μ'' is on the boundary of Φ.

6°. If $\Phi < \Psi$, μ'' is non-trivial and $\sigma_j = \beta(\Gamma_j)$, then σ_j contains a boundary edge of Ψ.

On the other hand, by Lemma 9, applied with $\Gamma = \Phi$, we obtain:

7°. If $\Phi < \Psi$ and σ_j is on the boundary of Φ, then it is also on the boundary of Ψ.

Using 6° and 7°, we obtain:

8°. If $\Phi < \Psi$, μ'' is non-trivial and $q \geqq 2$, then $\sigma_2\cdots\sigma_{q-1}$ is on the boundary of Ψ.

Indeed, consider the path $\kappa := \mu''^{-1}$. Applying 2°, 3° and 5° with Φ, Ψ, μ replaced by Ψ, Φ, κ, we obtain a factorization

$$\kappa = \kappa'\kappa''\kappa''' \tag{8}$$

such that

(α) κ' is a head of $\mathrm{RT}(\mathrm{o}(\kappa);\Psi)$;

(β) κ'''^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\kappa);\Psi')$;

(γ) if κ'' is non-trivial then κ'' is on the boundary of Ψ.

Since κ' and κ'''^{-1} are heads of transversals to Ψ, they do not contain boundary edges of Ψ.

Comparing (7) and (8), we see that

$$\sigma_1\sigma_2\cdots\sigma_{q-1}\sigma_q = \mu'' = \kappa^{-1} = \kappa'''^{-1}\kappa''^{-1}\kappa'^{-1}.$$

By 6° and 7°, σ_1 and σ_q contain boundary edges of Ψ; therefore κ'''^{-1} is a head of σ_1 and κ'^{-1} is a tail of σ_q. Then $\sigma_2\cdots\sigma_{q-1}$ is a subpath of κ''^{-1}; hence $\sigma_2\cdots\sigma_{q-1}$ is on the boundary of Ψ, as required.

9°. $\mathrm{pr}(\mu';\Phi) \in \Phi(2e_1)$ and $\mathrm{pr}(\mu''';\Phi) \in \Phi(2e_1)$.

Indeed, since by 2° μ' is a head of $\mathrm{RT}(\mathrm{o}(\mu);\Phi)$, it follows by Lemma 18 that $\mathrm{pr}(\mu';\Phi) = \mathrm{pr}(\mathrm{o}(\mu);\Phi)$ and then by Lemma 24(c) that $\mathrm{pr}(\mathrm{o}(\mu);\Phi) \in \Phi(2e_1)$. Similarly, we obtain

$$\mathrm{pr}(\mu'''; \Phi) = \mathrm{pr}(\mathrm{t}(\mu); \Phi) \in \Phi(2e_1),$$

as required.

10°. If $\Phi < \Psi$ and μ does not contain boundary edges of Ψ, then $\mathrm{pr}(\mu; \Phi) \in \Phi(4e_1)$.

Indeed, if μ'' is non-trivial, then by 6° and 7° μ contains boundary edges of Ψ. Therefore μ'' is trivial and then, by 9°, Lemma 16 and Lemma 7(d), $\mathrm{pr}(\mu; \Phi) = \mathrm{pr}(\mu'\mu'''; \Phi) \in (4e_1)$.

11°. If $\Psi < \Phi$, then $\mathrm{pr}(\mu'; \Psi) \in \Psi(4e_1)$ and $\mathrm{pr}(\mu'''; \Psi) \in \Psi(4e_1)$.

By Lemma 9, any edge e in μ' which is a boundary edge of Ψ is also a boundary edge of Φ. But by 2° μ' is a head of a transversal to Φ, hence it does not contain boundary edges of Φ. Therefore, neither does μ' contain boundary edges of Ψ. Applying 10° with Φ, Ψ, μ replaced by Ψ, Φ, μ'^{-1}, we see that $\mathrm{pr}(\mu'; \Psi) \in \Psi(4e_1)$. Similarly, $\mathrm{pr}(\mu'''; \Psi) \in \Psi(4e_1)$, as required.

12°. If $\Psi < \Phi$, μ'' is non-trivial and μ_0 is a head of μ'' such that μ_0 is a subpath of $\beta(\Pi)^{-1}$ for some $\Pi \in \mathscr{L}^1(\Psi)$, then $\mathrm{pr}(\mu'\mu_0; \Psi) \in \Psi(5e_1)$.

Indeed, $\mathrm{pr}(\mu_0; \Psi)$ is a subpath of $\alpha(\Pi)^{-1}$. Hence $\mathrm{pr}(\mu_0; \Psi) \in \Psi(e_1)$. Then, in view of 11°, $\mathrm{pr}(\mu'\mu_0; \Psi) \in \Psi(5e_1)$.

13°. Let $\Phi < \Psi$.

(1) $\mathrm{pr}(\mu'; \Psi) \in \Psi(4e_1)$ and $\mathrm{pr}(\mu'''; \Psi) \in \Psi(4e_1)$.

(2) Let μ'' be non-trivial. If $\sigma_1 = \beta(\Gamma_1)$ for some $\Gamma_1 \in \mathscr{L}^1(\Phi)$, then $\mathrm{pr}(\mu'\sigma_1; \Psi) \in \Psi(5e_1)$, and if $\sigma_q = \beta(\Gamma_q)$ for some $\Gamma_q \in \mathscr{L}^1(\Phi)$, then $\mathrm{pr}(\sigma_q\mu'''; \Psi) \in \Psi(5e_1)$.

Denote $\tau := \mu'^{-1}$ in case (1) and $\tau := \sigma_1^{-1}\mu'^{-1}$ in case (2). In view of Lemma 15(g) and Lemma 24(c), we have to show that in case (1) $\mathrm{pr}(\tau, \Psi) \in \Psi(4e_1)$ and in case (2) $\mathrm{pr}(\tau; \Psi) \in \Psi(5e_1)$.

Applying 2°, 3° and 5° with Φ, Ψ, μ replaced by Ψ, Φ, τ, we obtain a factorization

$$\tau = \tau'\tau''\tau''' \tag{9}$$

such that

(α) τ' is a head of $\mathrm{RT}(\mathrm{o}(\tau); \Psi)$;

(β) τ'''^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\tau); \Psi)$;

(γ) if τ'' is non-trivial then τ'' is on the boundary of Ψ.

Applying 9° with Φ, μ replaced by Ψ, τ we obtain

(δ) $\mathrm{pr}(\tau'; \Psi) \in \Psi(2e_1)$ and $\mathrm{pr}(\tau'''; \Psi) \in \Psi(2e_1)$.

If τ'' is trivial, then, by Lemma 16 and Lemma 7(d), $\mathrm{pr}(\tau; \Psi) = \mathrm{pr}(\tau'\tau'''; \Psi) \in \Psi(4e_1)$, as required. Assume now that τ'' is non-trivial. Then, in view of Lemma 16 and Lemma 7(d), (e),

(10) $$\mathrm{pr}(\tau;\Psi)=\mathrm{pr}(\tau';\Psi)\tau''\mathrm{pr}(\tau''';\Psi).$$

Since μ' is (by 2°) a head of $\mathrm{RT}(\mathrm{o}(\mu);\Phi)$, we can write $\mu'=\pi_1\pi_2$, where π_1 does not contain boundary edges of Φ^1 and π_2 (if non-trivial) is on the boundary of Φ^1. By 4°, σ_1 is on the boundary of Φ^1 and therefore in both cases (1) and (2), we can write

(11) $$\tau=\phi_1\phi_2$$

where ϕ_1 (if non-trivial) is on the boundary of Φ^1 and ϕ_2 does not contain boundary edges of Φ^1.

Since, by (γ), τ'' is on the boundary of $\Psi=\Psi^0$, it is also on the boundary of Φ^1, by Lemma 9. By (9) and (11), $\tau=\tau'\tau''\tau'''=\phi_1\phi_2$.

Since ϕ_2 does not contain boundary edges of Φ^1, we see that τ'' is a subpath of ϕ_1, and then τ' is a head of ϕ_1. Hence τ' is on the boundary of Φ^1. We have $\Phi<\Psi$ and therefore, by Lemma 9, τ' is on the boundary of Ψ^1. (See Fig. 57). By Definition 23, $\Psi^1=\mathrm{int}(C^1(\Psi))$ and, by Definition 25, $C^1(\Psi)=E^1(\Psi)$. By Lemma 13, $E'(\Psi)$ is an elementary map over Ψ. By (α), τ' is a head of $\mathrm{RT}(\mathrm{o}(\tau);\Psi)$. Therefore, by Definition 19, we obtain

(12) $$\mathrm{pr}(\tau';\Psi)=\mathrm{pr}(\mathrm{o}(\tau);\Psi)\in\Psi(e_1).$$

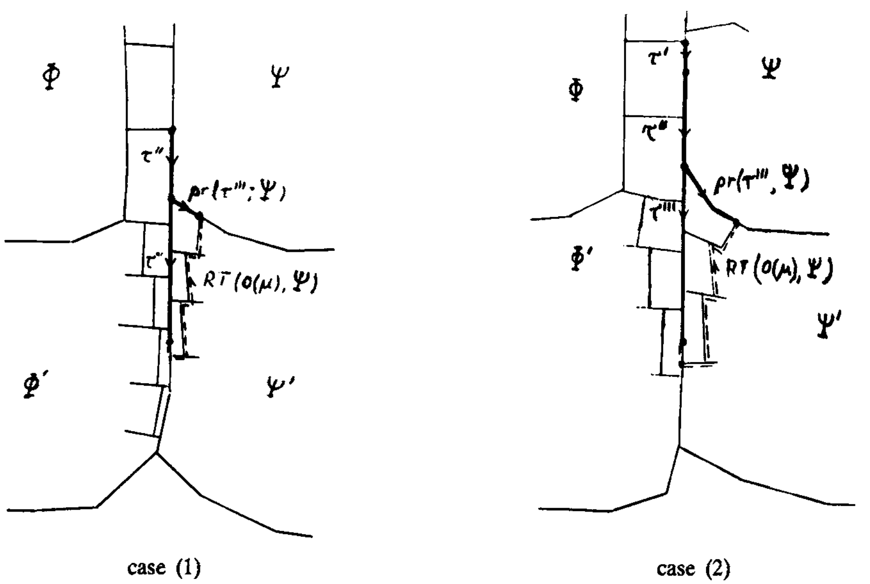


Fig. 57.

Consider the l.h.s. factorization of μ' in M, $\mu' = \lambda_1\lambda_2\cdots\lambda_r$, and let $\Lambda_1, \Lambda_2, \cdots, \Lambda_r$ be the corresponding sequence of regions. Denote

$$l_j := \mathrm{d}(\Lambda_j, \Phi), \qquad 1 \leqq j \leqq r.$$

Since μ' is a head of $\mathrm{RT}(\mathrm{o}(\mu); \Phi)$, it follows from Lemma 17(f) that

$$l_1 > l_2 > \cdots > l_{r-1} > l_r > 0,$$

hence $l_j \geqq 2$, $j = 1, 2, \cdots, r-1$.

Therefore $\lambda_1\lambda_2\cdots\lambda_{r-1}$ does not contain boundary edges of Φ^1. Thus by Lemma 9, $\lambda_1\lambda_2\cdots\lambda_{r-1}$ does not contain boundary edges of Ψ. The path μ' is a head of τ^{-1}; hence $\lambda_{r-1}^{-1}\lambda_{r-2}^{-1}\cdots\lambda_2^{-1}\lambda_1^{-1}$ is a tail of τ. By (γ), τ'' (assumed non-trivial) is on the boundary of Ψ.

In case (1) we have $\tau = \mu'^{-1} = \lambda_r^{-1}\lambda_{r-1}^{-1}\cdots\lambda_2^{-1}\lambda_1^{-1}$ and then τ'' is a subpath of λ_r^{-1}. The path τ'' is then on the common boundary of Ψ and Λ_r, where $\mathrm{d}(\Lambda_r, \Phi) = l_r > 0$. By Lemma 8(c), $\Lambda_r \in \mathcal{T}_1$; hence, by Definition 30,

$$\tau'' \in \Psi(e_1). \tag{13}$$

In case (2) we have $\tau = \sigma_1^{-1}\mu'^{-1} = \sigma_1^{-1}\lambda_r^{-1}\lambda_{r-1}^{-1}\cdots\lambda_2^{-1}\lambda_1^{-1}$, and then τ'' is a subpath of $\sigma_1^{-1}\lambda_r^{-1}$. The path σ_1^{-1} is on the boundary of $\Gamma_1 \in \mathcal{L}^1(\Phi) \subseteq \mathcal{T}_1$ and λ_r^{-1} is on the boundary of $\Lambda_r \in \mathcal{T}_1$. Therefore,

$$\tau'' \in \Psi(2e_1). \tag{14}$$

In view of (10), in case (1) it follows from (δ), (12) and (13) that $\mathrm{pr}(\tau; \Psi) \in \Psi(4e_1)$, and in case (2) from ($\delta$), (12) and (14) that $\mathrm{pr}(\tau; \Phi) \in \Phi(5e_1)$. The remaining assertions of 13° are verified similarly.

All the assertions of the proposition have now actually been proved. Indeed, we have a factorization (5) which, by 2°, 3° and 9°, possesses properties (a), (b), (c). Property (d) follows from 11° and 13°, (1). If μ'' is non-trivial and $\Phi < \Psi$, then we take (6) to be the l.h.s. factorization of μ'' in M. Then by 4°, $q = h$ and $\mu_j = \sigma_j$ for $j = 1, \cdots, h$. Properties (e) and (h) follow now from 4°, (f) follows from 6°, 7° and 8°, (g) is satisfied by the construction of (6) and (i) follows from 13°.

If μ'' is non-trivial and $\Psi < \Phi$, then we take (6) to be the r.h.s. factorization of μ'' in M and let $\Pi_1, \Pi_2, \cdots, \Pi_h$ be the corresponding sequence of regions. Then property (e′) follows from 5°, (f′) follows from (e′) by Lemma 9, and (g′) is true by the construction of (6). Since, by (f′), μ'' is on the boundary of Ψ^1, it follows that $0 \leqq \mathrm{d}(\Pi_j, \Psi) \leqq 1$ for any j, $1 \leqq j \leqq h$. If for some j, $1 \leqq j \leqq h$, μ_j is not on the boundary of Ψ, then $\mathrm{d}(\Pi_j, \Psi) = 1$. Therefore, if $1 < j < h$, then $\mathrm{d}(\Pi_{j-1}, \Psi) \leqq 1 = \mathrm{d}(\Pi_j, \Psi)$ and $\mathrm{d}(\Pi_{j+1}, \Psi) \leqq 1 = \mathrm{d}(\Pi_j, \Psi)$. Thus, by Lemma 17(d), $\mu_j = \beta(\Pi_j)^{-1}$, where $\Pi_j \in \mathcal{L}^1(\Psi)$. Thus, property (h′) also holds. The first assertion of (i′) follows from 12°. The second assertion of (i′) can be proved in similar fashion.

The proposition is proved.

4.4. *An "area theorem" for ordered 2-ranked maps.*

PROPOSITION 3. *Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2\}, <)$ be an ordered 2-ranked map satisfying conditions* (SC), D(6; 1), D(8). *Let the subset $\mathcal{U}$ of $\mathcal{T}_1$ be defined by*

$$\mathcal{U} := \{\Phi \mid \Phi \in \mathcal{T}_1, \operatorname{ind}(\Phi) \leqq 2e_1 + 2e_2\}.$$

Assume that M is connected and simply-connected, and let ω be a boundary cycle of M. Then $\operatorname{card}(\mathcal{T}_1 \setminus \mathcal{U})$ *is effectively bounded in terms of* $\operatorname{card}(\mathcal{T}_2)$ *and the length of ω.*

REMARK. The assumptions of connectedness and simply-connectedness can be omitted; we have only to consider instead of ω a system of boundary cycles describing $\operatorname{bd}(M)$.

PROOF. Consider the derived map M'. Since $\mathcal{M}$ satisfied (SC), for any region Φ' in M', $\operatorname{clos}(\Phi')$ is simply-connected. Let $h = \operatorname{card}(\mathcal{T}_2)$. For any two regions Φ' and Ψ' in M', the intersection of their boundaries, $\operatorname{bd}(\Phi') \cap \operatorname{bd}(\Psi')$ has no more than $h-1$ connected components. Therefore we can find paths $\mu_1, \mu_2, \cdots, \mu_k$ such that

1°. Each μ_1 is a p.o.b.p. of some regions Φ'_i and a n.o.b.p. of some region Ψ'_i, $1 \leqq i \leqq k$.

2°. Each (non-oriented) edge of M' belongs exactly to one of the paths $\mu_1, \mu_2, \cdots, \mu_k, \omega$.

3°. $$k \leqq \frac{h(h-1)}{2}(h-1) \leqq \frac{1}{2} h^3.$$

Substituting, if necessary, μ_i by μ_i^{-1}, we may assume without loss of generality that

4°. $\Phi_i < \Psi_i$ in the linear order on $\mathcal{T}_2$, $1 \leqq i \leqq k$.

Applying Proposition 2 to the path μ_i, we obtain a factorization

$$\mu_i = \mu'_i \mu''_i \mu'''_i \tag{15}$$

and, if μ''_i is non-trivial, a further factorization

$$\mu''_i = \mu_{i1} \mu_{i2} \cdots \mu_{ih(i)} \tag{16}$$

with the properties described in Proposition 2.

If μ''_i is non-trivial, then by Proposition 2(g) (16) is the l.h.s. factorization of μ''_i. Let

(17) $$\Gamma_{i1}, \Gamma_{i2}, \cdots, \Gamma_{ih(i)}$$

be the corresponding sequence of regions. We consider the set of regions

(18) $$\mathcal{U}_i = \{\Gamma_{ij} \mid 1 < j < h(i), \mathrm{d}(\Gamma_{ij}, \Phi_i) = 1\}.$$

For any $\Phi \in \mathcal{T}_2$, there may be some values of i such that $\Phi_i = \Phi$ $(1 \leqq i \leqq k)$. We denote $\mathcal{U}(\Phi)$ the union of all sets $\mathcal{U}_i$ such that $\Phi_i = \Phi$. Let $Q(\Phi)$ be the regular submap of M such that

$$\mathrm{Reg}(Q(\Phi)) = \{\Phi\} \cup \mathcal{U}(\Phi).$$

Since, by (18), $\mathcal{U}(\Phi) \subseteq \mathcal{L}^1(\Phi)$, we have $C^0(\Phi) \subseteq Q(\Phi) \subseteq C^1(\Phi)$.

Therefore, by Lemma 11 and (SC), $\mathrm{supp}(Q(\Phi))$ is connected and simply-connected.

We denote $\tilde{M}$ the map obtained from M by deleting $\mathrm{int}(Q(\Phi))$ for all $\Phi \in \mathcal{T}_2$.

5°. $\mathcal{U}(\Phi) \subseteq \mathcal{U}$ for all $\Phi \in \mathcal{T}_2$.

Indeed, consider some region Γ_{ij}, $1 < j < h(i)$, such that $\mathrm{d}(\Gamma_{ij}, \Phi_i) = 1$. According to Proposition 2(f), (h), $\beta(\Gamma_{ij}) = \mu_{ij}$ is on the boundary of Ψ_i, therefore $\mathrm{ind}(\Gamma_{ij}, \beta(\Gamma_{ij})) = e_2$ and then, by Lemma 22(b), $\mathrm{ind}(\Gamma_{ij}) \leqq 2e_1 + 2e_2$. Hence, $\Gamma_{ij} \in \mathcal{U}$. In view of (18), $\mathcal{U}_i \subseteq \mathcal{U}$ and then $\mathcal{U}(\Phi) \subseteq \mathcal{U}$ for all $\Phi \in \mathcal{T}_2$, as required.

In view of 5°, it is enough to show that the number of regions of $\tilde{M}$ is effectively bounded in terms of $h = \mathrm{card}(\mathcal{T}_2)$ and $|\omega|$.

Let

(19) $$\Lambda_{i1}, \Lambda_{i2}, \cdots, \Lambda_{il(i)}$$

be the sequence of regions corresponding to the l.h.s. factorization of μ_i in M, and

(20) $$\mathrm{P}_{i1}, \mathrm{P}_{i2}, \cdots, \mathrm{P}_{ir(i)}$$

the sequence of regions corresponding to the r.h.s. factorization of μ_i in M. Consider the set of regions $\mathcal{W} \subseteq \mathcal{T}_1$ defined by

(21) $$\mathcal{W} = \{\Pi \mid \Pi \in \mathcal{T}_1, \mathrm{ind}(\Pi) = d_0 r_0 + d_1 e_1 + d_2 e_2, d_2 \geqq 2\}.$$

6°. $\mathcal{W} \subseteq \{\Lambda_{ij} \mid 1 \leqq j \leqq l(i), 1 \leqq i \leqq k\} \cup \{\mathrm{P}_{ij} \mid 1 \leqq j \leqq r(i), 1 \leqq i \leqq k\}$.

Indeed, let $\Pi \in \mathcal{W}$. Then $\Pi \in \mathcal{T}_1$. By Lemma 8(c), for some $\Phi \in \mathcal{T}_2$, $\Pi \in \mathcal{L}(\Phi)$. Let $\mathrm{ind}(\Pi) = d_0 e_0 + d_1 e_1 + d_2 e_2$. Since $d_2 \geqq 2 > 0$, there is at least one region $\Psi \in \mathcal{T}_2$ such that $\mathrm{d}(\Pi, \Psi) = 1$. Then by Definition 21, $\Pi \in \mathcal{L}^1(\Phi)$. By Lemma 6,

$\mathrm{bd}(\Pi) \cap \mathrm{bd}(\Phi)$ is connected and described by $\alpha(\Pi)$. Therefore, in view of $d_2 \geqq 2$, there is a region $\Phi_0 \in \mathcal{T}_2$, $\Phi_0 \neq \Phi$, such that Π and Φ_0 have a common boundary edge e. According to Definition 24, $\Pi \subseteq \Phi'$ and $\Phi_0 \subseteq \Phi_0'$. Therefore the edge e is on the common boundary if Φ' and Φ_0'. Then, according to 2°, for some i, $1 \leqq i \leqq k$, the path μ_i contains e. Then the region Π appears either in the sequence (19) or in the sequence (20) for the same value of i.

This proves our assertion.

7°. (α) $\mathrm{card}(\mathcal{W} \setminus \mathcal{U}_i) \cap \{\Lambda_{i1}, \Lambda_{i2}, \cdots, \Lambda_{il(i)}\}) \leqq 4$;

(β) $\mathrm{card}(\mathcal{W} \cap \{\mathrm{P}_{i1}, \mathrm{P}_{i2}, \cdots, \mathrm{P}_{ir(i)}) \leqq 2$, $1 \leqq i \leqq k$.

Consider the sequence of regions

$$\Sigma_{i1}, \Sigma_{i2}, \cdots, \Sigma_{is(i)} \tag{22}$$

which corresponds to the l.h.s. factorization of μ_i' in M. Let $l_j = \mathrm{d}(\Sigma_{ij}, \Phi_i)$, $1 \leqq j \leqq s(i)$. By Proposition 2(a), μ_i' is a head of $\mathrm{RT}(\mathrm{o}(\mu); \Phi)$. Then by Lemma 17(f),

$$l_1 > l_2 > \cdots > l_{s(i)} > 0.$$

Since $\Sigma_{ij} \in \mathcal{L}(\Phi_i)$, by Definition 21, $\mathrm{d}(\Sigma_{ij}, \Psi) \geqq \mathrm{d}(\Sigma_{ij}, \Phi_i)$ for any $\Psi \in \mathcal{T}_2$. Since $l_j > 1$ for $j = 1, 2, \cdots, s(i) - 1$, we obtain that for $\Sigma_{i1}, \Sigma_{i2}, \cdots, \Sigma_{is(i)-1}$ there is no region Ψ in $\mathcal{T}_2$ such that $\mathrm{d}(\Sigma_{ij}, \Psi) = 1$, $1 \leqq j \leqq s(i) - 1$. Therefore,

$$\mathcal{W} \cap \{\Sigma_{i1}, \Sigma_{i2}, \cdots, \Sigma_{is(i)}\} \subseteq \{\Sigma_{is(i)}\}.$$

Similarly, let

$$\Pi_{i1}, \Pi_{i2}, \cdots, \Pi_{ip(i)} \tag{23}$$

be the sequence of regions which corresponds to the l.h.s. factorization of μ_i''' in M. Then

$$\mathcal{W} \cap \{\Pi_{i1}, \Pi_{i2}, \cdots, \Pi_{ip(i)}\} \subseteq \{\Pi_{i1}\}.$$

By the construction of $\mathcal{U}_i$, we have

$$(\mathcal{W} \setminus \mathcal{U}_i) \cap \{\Gamma_{i1}, \Gamma_{i2}, \cdots, \Gamma_{ih(i)}\} \subseteq \{\Gamma_{i1}, \Gamma_{ih(i)}\}.$$

In view of (15),

$$\{\Lambda_{i1}, \Lambda_{i2}, \cdots, \Lambda_{il(i)}\} = \{\Sigma_{i1}, \cdots, \Sigma_{is(i)}\} \cup \{\Gamma_{i1}, \cdots, \Gamma_{ih(i)}\} \cup \{\Pi_{i1}, \cdots, \Pi_{ip(i)}\}.$$

Therefore

$$(\mathcal{W} \setminus \mathcal{U}_i) \cap \{\Lambda_{i1}, \Lambda_{i2}, \cdots, \Lambda_{il(i)}\} \subseteq \{\Sigma_{is(i)}, \Gamma_{i1}, \Gamma_{ih(i)}, \Pi_{i1}\}.$$

This proves (α). To prove (β), we consider the path $\kappa_i := \mu_i^{-1}$, which is a p.o.b.p. of Ψ_i and a n.o.b.p. of Φ_i. Applying Proposition 2 with Φ, Ψ, μ replaced by Ψ_i, Φ_i, κ_i, we obtain a factorization $\kappa_i = \kappa_i' \kappa_i'' \kappa_i'''$ with the properties described in Proposition 2. In particular, since $\Phi_i < \Psi_i$, we obtain by (e′) that if κ_i'' is non-trivial, then it is on the boundary of Ψ_i. Considering the sequences of regions for the l.h.s. factorizations of κ_i' and κ_i''' in M, we conclude that both of them contain at most one region from $\mathcal{W}$. Therefore $\mathcal{W} \cap \{P_{i1}, P_{i2}, \cdots, P_{ir(i)}\}$ contains at most 2 regions, as required.

Let $\mathcal{U}_0 = \bigcup_{\Phi \in \mathcal{T}_2} \mathcal{U}(\Phi)$. Using 3°, 6° and 7°, we obtain

8°. $\mathrm{Card}(\mathcal{W} \setminus \mathcal{U}_0) \leqq 6k \leqq 3h^3$.

Let $\mathcal{B} \subseteq \mathcal{T}_1$ be the set of all boundary regions $\Pi \in \mathcal{T}_1$ (i.e. consists of all regions $\Pi \in \mathcal{T}_1$ such that $\mathrm{bd}(\Pi) \cap \mathrm{bd}(M)$ contains at least one edge). Then, obviously,

9°. $\mathrm{card}(\mathcal{B}) \leqq |\omega|$.

10°. Let $\Gamma \in \mathcal{U}_0$, $\Pi \in \mathcal{T}_1$ and $\mathrm{d}(\Gamma, \Pi) = 1$. Then $\Pi \in \mathcal{W}$.

Indeed, by the definition of $\mathcal{U}_0$, we have $\Gamma = \Gamma_{ij}$ for some i, j $(1 \leqq i \leqq k$, $1 < j < h(i))$. By Lemma 6 and Definition 26, $\alpha(\Gamma_{ij})^{-1}\gamma(\Gamma_{ij})^{-1}\beta(\Gamma_{ij})\delta(\Gamma_{ij})$ is a p.o.b.p. of Γ_{ij}. Here $\alpha(\Gamma_{ij})$ is on the common boundary of Γ_{ij} and $\Phi_i \in \mathcal{T}_2$, while by Proposition 2(h), $\beta(\Gamma_{ij})$ is on the common boundary of Γ_{ij} and $\Psi_i \in \mathcal{T}_2$. According to Lemma 6(d), if $\gamma(\Gamma_{ij})$ is non-trivial then $\gamma(\Gamma_{ij}) = \delta(\Gamma_{ij-1})$, where $\Gamma_{ij-1} \in \mathcal{L}^1(\Phi)$. Then, of course, $\mathrm{d}(\Gamma_{ij}, \Gamma_{ij-1}) = 1$ and $\Gamma_{ij-1} \in \mathcal{T}_1$. The region Γ_{ij-1} has common boundary edges with Φ_i and Ψ_i; therefore $\Gamma_{ij-1} \in \mathcal{W}$.

Similarly, if $\delta(\Gamma_{ij})$ is non-trivial then for Γ_{ij+1} we have $\delta(\Gamma_{ij}) = \gamma(\Gamma_{ij+1})$, and $\Gamma_{ij+1} \in \mathcal{W}$.

This proves our assertion.

Let $\Pi \in \mathcal{T}_1 \setminus (\mathcal{W} \cup \mathcal{B})$ and let $\mathrm{ind}(\Pi) = d_0e_0 + d_1e_1 + d_2e_2$. Then $d_0 = 0$ because Π is an inner region of M and $d_2 \leqq 1$ because $\Pi \notin \mathcal{W}$. Since $\mathcal{M}$ satisfies D(8) and D(6; 1), we obtain $d_1 \geqq 7$. By 10°, for each $\Gamma \in \mathcal{T}_1$ such that $\mathrm{d}(\Pi, \Gamma) = 1$ we have $\Gamma \notin \mathcal{U}_0$. Hence:

11°. In $\tilde{M}$, for each region $\Pi \in \mathcal{T}_1 \setminus (\mathcal{W} \cup \mathcal{B})$, $d_{\tilde{M}}(\Pi) \geqq 7$, where $d_{\tilde{M}}(\Pi)$ denotes the *index* of Π in $\tilde{M}$.

The map $\tilde{M}$ is connected and has $h = \mathrm{card}(\mathcal{T}_2)$ holes (i.e. bounded connected components of $\mathrm{compl}(\tilde{M})$). We apply to $\tilde{M}$ formula (3.1) from [1], p. 243, with $p = 3$, $q = 6$. Since $\Sigma[3 - d(v)] \leqq 0$, we obtain

$$3(1-h) \leqq \frac{1}{2} \sum (6 - d_{\tilde{M}}(\Pi))$$

where the sum is taken over all regions Π in $\tilde{M}$.

Since $\operatorname{Reg}(\tilde{M})=(\mathcal{T}_1\setminus(\mathcal{W}\cup\mathcal{B}))\cup((\mathcal{W}\cup\mathcal{B})\setminus\mathcal{U}_0)$, we can write

$$\Sigma_1(6-d_{\tilde{M}}(\Pi))\geqq 6(1-h)-\Sigma_2(6-d_{\tilde{M}}(\Pi))$$

where in Σ_1 the sum is taken over all regions $\Pi\in\mathcal{T}_1\setminus(\mathcal{W}\cup\mathcal{B})$ and in Σ_2 the sum is taken over all regions $\Pi\in(\mathcal{W}\cup\mathcal{B})\setminus\mathcal{U}_0$.

By 8° and 9°, $\operatorname{card}((\mathcal{W}\cup\mathcal{B})\setminus\mathcal{U}_0)\leqq 3h^3+|\omega|$, hence

$$\Sigma_2(6-d_{\tilde{M}}(\Pi))\leqq 18h^3+6|\omega|.$$

On the other hand, in view of 11°, we have

$$\Sigma_1(6-d_{\tilde{M}}(\Pi))\leqq -\operatorname{card}(\mathcal{T}_1\setminus(\mathcal{W}\cup\mathcal{B})).$$

We obtain

$$\operatorname{card}(\mathcal{T}_1\setminus(\mathcal{W}\cup\mathcal{B}))\leqq 18h^3+6h+6|\omega|-6.$$

Therefore

$$\begin{aligned}\operatorname{card}(\mathcal{T}_1\setminus\mathcal{U})&\leqq\operatorname{card}(\mathcal{T}_1\setminus\mathcal{U}_0)\leqq\operatorname{card}(\mathcal{T}_1\setminus(\mathcal{W}\cup\mathcal{B}))+\operatorname{card}((\mathcal{W}\cup\mathcal{B})\setminus\mathcal{U}_0)\\&\leqq 21h^3+6h+7|\omega|-6.\end{aligned}$$

The proposition is proved.

§5. Ordered n-ranked maps

5.1. *Conditions* (SC_i). Given an ordered 2-ranked map, we defined condition (SC); when this condition was satisfied, we constructed a derived map. We now extend this idea to arbitrary n.

More precisely, we shall introduce a family of conditions (SC_i), $0\leqq i\leqq n-1$, where each (SC_j) is stronger than (SC_{j-1}) and for any ordered n-ranked map $\mathcal{M}=(M,\{\mathcal{T}_1,\cdots,\mathcal{T}_n\},<)$ (see Definition 12) satisfying (SC_i) we shall construct a sequence

$$\mathcal{M}^{(0)}=\mathcal{M},\mathcal{M}^{(1)},\mathcal{M}^{(2)},\cdots,\mathcal{M}^{(i)} \tag{1}$$

where $\mathcal{M}^{(j)}$ is an ordered $(n-j)$-ranked map.

The inductive definition is as follows:

(1) $\mathcal{M}$ satisfies (SC_0) if, for any region Φ in M, clos(Φ) is simply-connected.

(2) Assume that (SC_{i-1}) is defined; let $\mathcal{M}$ satisfy (SC_{i-1}) and let $\mathcal{M}^{(0)}=\mathcal{M},\mathcal{M}^{(1)},\mathcal{M}^{(2)},\cdots,\mathcal{M}^{(i-1)}$ be the corresponding sequence. $\mathcal{M}^{(i-1)}$ is an ordered $(n-i+1)$-ranked map. We can write

$$\mathcal{M}^{(i-1)}=(M^{(i-1)},\{\mathcal{T}_i^{(i-1)},\mathcal{T}_{i+1}^{(i-1)},\cdots,\mathcal{T}_n^{(i-1)}\},<),$$

where $\mathcal{T}_j^{(i-1)}$ is the set of regions of $M^{(i-1)}$ of rank $j-i+1$. Form an ordered 2-ranked map

(2) $$\tilde{\mathcal{M}}^{(i-1)} := (M^{(i-1)}, \{\mathcal{T}_i^{(i-1)}, \mathcal{T}_{i+1}^{(i-1)} \cup \cdots \cup \mathcal{T}_n^{(i-1)}\}, <)$$

changing the rank of all regions Φ with $\operatorname{rank}(\Phi) > 1$ to rank 2. We shall say that $\mathcal{M}$ satisfies (SC_i) if ($\mathcal{M}$ satisfies (SC_{i-1}) and) the ordered 2-ranked map $\tilde{\mathcal{M}}^{(i-1)}$ satisfies condition (SC). If this is the case, $\mathcal{M}^{(i)}$ is constructed as follows:

(3) $$\mathcal{M}^{(i)} := (M^{(i)}, \{\mathcal{T}_{i+1}^{(i)}, \mathcal{T}_{i+2}^{(i)}, \cdots, \mathcal{T}_n^{(i)}\}, <)$$

where $M^{(i)}$ is the derived map of $\tilde{\mathcal{M}}^{(i-1)}$, $\mathcal{T}_j^{(i)}$ is given by

$$\mathcal{T}_j^{(i)} := \{\Phi' \mid \Phi \in \mathcal{T}_j^{(i-1)}\}, \qquad i+1 \leqq j \leqq n,$$

and the order relation "$<$" on $\mathcal{T}_{i+2}^{(i)} \cup \mathcal{T}_{i+3}^{(i)} \cup \cdots \cup \mathcal{T}_n^{(i)}$ is induced from $\mathcal{T}_{i+2}^{(i-1)} \cup \cdots \cup \mathcal{T}_n^{(i-1)}$ by the mapping $\Phi \mapsto \Phi'$. Since $M^{(i)}$ is a derived map, it is normalized by Lemma 12 and regular (see Definitions 6, 12 and 24); $\operatorname{int}(M^{(i)}) = \operatorname{int}(M)$ is connected; $\mathcal{T}_n$ corresponds in one-to-one fashion

$$\Phi \mapsto \Phi' \mapsto \Phi'' \mapsto \cdots \mapsto \Phi^{(i)}$$

with $\mathcal{T}_n^{(i)}$; therefore $\mathcal{T}_n^{(i)}$ is non-empty. For Φ, Ψ in $M^{(i)}$ with $1 < \operatorname{rank}(\Phi) < \operatorname{rank}(\Psi)$ we have $\Phi < \Psi$. Therefore, according to Definition 12, $\mathcal{M}^{(i)}$ is an ordered $(n-i)$-ranked map. The sequence

$$\mathcal{M}^{(0)} = \mathcal{M}, \mathcal{M}^{(1)}, \cdots, \mathcal{M}^{(i-1)}, \mathcal{M}^{(i)}$$

is thus constructed.

If $\Phi \in \operatorname{Reg}(M)$ and $\operatorname{rank}(\Phi) > i$, we let $\Phi^{(i)}$ denote the region of $M^{(i)}$ corresponding to Φ under the mapping

(4) $$\Phi \mapsto \Phi' \mapsto (\Phi')' = \Phi'' = \Phi^{(2)} \mapsto \cdots \mapsto \Phi^{(i-1)} \mapsto (\Phi^{(i-1)})' = \Phi^{(i)}.$$

For example, let, in Fig. 58, $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4\}, <)$ be an ordered 4-ranked map, where $\mathcal{T}_2 = \{\Phi, \Psi\}$, $\mathcal{T}_3 = \{\Gamma\}$, $\mathcal{T}_4 = \{\Delta\}$ and $\Phi < \Psi < \Gamma < \Delta$. $\mathcal{M}$ satisfies (SC_3) and the sequence $\mathcal{M}^{(0)}, \mathcal{M}^{(1)}, \mathcal{M}^{(2)}, \mathcal{M}^{(3)}$ is as shown in Fig. 58. We have $\mathcal{T}_2^{(1)} = \{\Phi^{(1)}, \Psi^{(1)}\}$, $\mathcal{T}_3^{(1)} = \{\Gamma^{(1)}\}$, $\mathcal{T}_4^{(1)} = \{\Delta^{(1)}\}$, $\mathcal{T}_3^{(2)} = \{\Gamma^{(2)}\}$, $\mathcal{T}_4^{(2)} = \{\Delta^{(2)}\}$, $\mathcal{T}_4^{(3)} = \{\Delta^{(3)}\}$.

5.2. *Transversals and projections in ordered n-ranked maps.* Let $\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n\}, <)$ be an ordered n-ranked map satisfying condition (SC_i) for some i, $0 \leqq i < n$. Then we have the sequence (1) defined in the previous section. Let $\Phi \in \operatorname{Reg}(M)$, $\operatorname{rank}(\Phi) > i$, and let μ be a boundary path of $\Phi^{(i)}$. By the construction of $\mathcal{M}^{(i)}$, $M^{(i)}$ is the derived map of $\tilde{\mathcal{M}}^{(i-1)}$; we can thus speak of

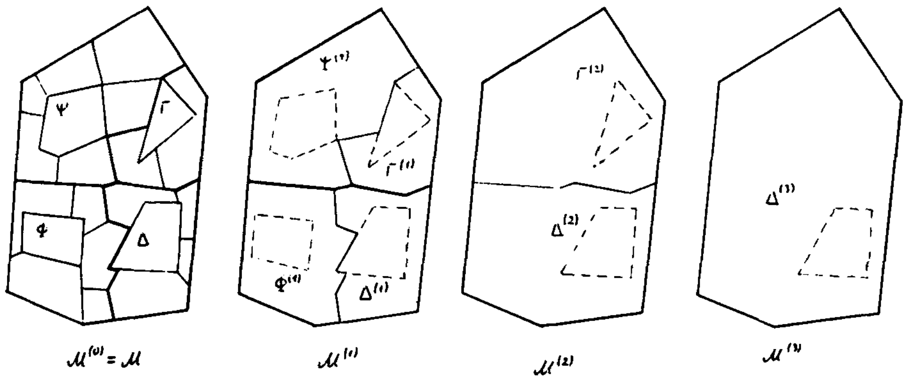


Fig. 58.

the projection $\mathrm{pr}_{\mathcal{M}^{(i-1)}}(\mu;\Phi^{(i-1)})$, which is a boundary path of $\Phi^{(i-1)}$. We can now take its projection to $\Phi^{(i-2)}$, and so on, until we obtain a boundary path of Φ which we call the projection of μ to Φ. Similarly, we define the right and left projections of μ to Φ, the shadow of μ with respect to Φ and right and left transversals and projections of a vertex $v \in \mathrm{bd}(\Phi^{(i)})$. More precisely, we have the following definition.

DEFINITION 32. Let $\Phi \in \mathrm{Reg}(M)$, $\mathrm{rank}(\Phi) > i$, let $v \in \mathrm{bd}(\Phi^{(i)})$. For $h = i, i-1, \cdots, 1, 0$, the left and right projections $\mathrm{lpr}_{\mathcal{M}}(v;\Phi^{(h)})$, or simply $\mathrm{lpr}(v;\Phi^{(h)})$, and $\mathrm{rpr}_{\mathcal{M}}(v;\Phi^{(h)})$, or $\mathrm{rpr}(v;\Phi^{(h)})$ from v to $\Phi^{(h)}$, and the left and right transversals $\mathrm{LT}_{\mathcal{M}}(v;\Phi^{(h)})$ or $\mathrm{LT}(v;\Phi^{(h)})$ and $\mathrm{RT}_{\mathcal{M}}(v;\Phi^{(h)})$ or $\mathrm{RT}(v;\Phi^{(h)})$ are defined recursively, as follows:

(5) $\mathrm{lpr}(v;\Phi^{(i)}) := v, \quad \mathrm{lpr}(v;\Phi^{(l-1)}) := \mathrm{lpr}_{\mathcal{M}^{(l-1)}}(\mathrm{lpr}(v;\Phi^{(l)});\Phi^{(l-1)}),$

(6) $\mathrm{lpr}(v;\Phi^{(i)}) := v, \quad \mathrm{rpr}(v;\Phi^{(l-1)}) := \mathrm{rpr}_{\mathcal{M}^{(l-1)}}(\mathrm{rpr}(v;\Phi^{(l)});\Phi^{(l-1)}),$

(7) $\mathrm{LT}(v;\Phi^{(i)}) := v, \quad \mathrm{LT}(v;\Phi^{(l-1)}) := \mathrm{LT}(v;\Phi^{(l)})\mathrm{LT}_{\mathcal{M}^{(l-1)}}(\mathrm{lpr}(v;\Phi^{(l)});\Phi^{(l-1)}),$

(8) $\mathrm{RT}(v;\Phi^{(i)}) := v, \quad \mathrm{RT}(v;\Phi^{(l-1)}) := \mathrm{RT}(v;\Phi^{(l)})\mathrm{RT}_{\mathcal{M}^{(l-1)}}(\mathrm{rpr}(v;\Phi^{(l)});\Phi^{(l-1)}),$

where $1 \leqq l \leqq i$.

Let μ be a boundary path of $\Phi^{(i)}$. The left and right projections $\mathrm{lpr}_{\mathcal{M}}(\mu;\Phi^{(h)})$, or simply $\mathrm{lpr}(\mu;\Phi^{(h)})$ and $\mathrm{rpr}_{\mathcal{M}}(\mu;\Phi^{(h)})$ or $\mathrm{rpr}(\mu;\Phi^{(h)})$ of μ to $\Phi^{(h)}$, the projection $\mathrm{pr}_{\mathcal{M}}(\mu;\Phi^{(h)})$ or $\mathrm{pr}(\mu;\Phi^{(h)})$ of μ to $\Phi^{(h)}$ and the shadow $\mathrm{S}_{\mathcal{M}}(\mu,\Phi^{(h)})$ or $\mathrm{S}(\mu;\Phi^{(h)})$ of μ with respect to $\Phi^{(h)}$ are defined recursively, as follows:

(9) $\mathrm{lpr}(\mu;\Phi^{(i)}) := \mu, \quad \mathrm{lpr}(\mu;\Phi^{(l-1)}) := \mathrm{lpr}_{\mathcal{M}^{(l-1)}}(\mathrm{lpr}(\mu;\Phi^{(l)});\Phi^{(l-1)}),$

(10) $\mathrm{rpr}(\mu;\Phi^{(i)}):=\mu,\quad \mathrm{rpr}(\mu;\Phi^{(l-1)}):=\mathrm{rpr}_{\mathcal{M}^{(i-1)}}(\mathrm{rpr}(\mu;\Phi^{(l)});\Phi^{(l-1)}),$

(11) $\mathrm{pr}(\mu;\Phi^{(i)}):=\mu,\quad \mathrm{pr}(\mu;\Phi^{(l-1)}):=\mathrm{pr}_{\mathcal{M}^{(i-1)}}(\mathrm{pr}(\mu;\Phi^{(l-1)});\Phi^{(l-1)}),$

$S(\mu;\Phi^{(i)})$ consists of the edges and vertices of μ and

(12) $S(\mu;\Phi^{(l-1)}):=S(\mu;\Phi^{(l)})\cup S_{(i-1)}(\mathrm{pr}(\mu;\Phi^{(l)});\Phi^{(l-1)})$

where $1\leqq l\leqq i$.

As an immediate consequence of the definitions we have:

LEMMA 26. *Let $\mathcal{M}$ be an ordered n-ranked map satisfying condition* (SC_l) *for some l, $0\leqq l<n$. Let Φ be a region in M such that* $\mathrm{rank}(\Phi)>l$.

(a) *Lemmas* 14 *and* 15 *remain valid when Φ^l is replaced by $\Phi^{(l)}$ and Φ^k is replaced by $\Phi^{(k)}$.*

(b) *Let $k\leqq l$; let $\mu=\mu_1\mu_2$ be a non-trivial p.o.b.p. of $\Phi^{(l)}$. Parts* (a), (b), (c), (d), (e), (f) *of Lemma* 7 *remain valid when Φ is replaced by $\Phi^{(k)}$.*

(c) *Lemma* 18 *remains valid when the condition $\Phi\in\mathcal{T}_2$ is omitted, Φ^l is replaced by $\Phi^{(l)}$ and Φ^k is replaced by $\Phi^{(k)}$.*

5.3. *Submaps.* Let $\mathcal{M}=(M,\{\mathcal{T}_1,\mathcal{T}_2,\cdots,\mathcal{T}_n\},<)$ be an ordered n-ranked map satisfying condition (SC_i) for some i, $0\leqq i\leqq n$. Let N be a regular submap of M such that $\mathrm{int}(N)$ is connected. Denote $\mathcal{U}_j:=\mathcal{T}_j\cap\mathrm{Reg}(N)$. Let m be maximal such that $\mathcal{U}_m\neq\varnothing$. The linear order "$<$" on $\mathcal{T}_2\cup\cdots\cup\mathcal{T}_n$ induces a linear order on $\mathcal{U}_2\cup\cdots\cup\mathcal{U}_m$, which we again denote by "$<$". Then, by Definition 12, $\mathcal{N}=(N,\{\mathcal{U}_1,\mathcal{U}_2,\cdots,\mathcal{U}_m\},<)$ is an ordered m-ranked map. The following definition extends Definition 28.

DEFINITION 33. *k-submaps.* Let $k\leqq i$. Let Q be a submap of M. We call Q a k-submap (of $\mathcal{M}$) it there is a subset $\mathcal{W}$ of $\mathcal{T}_{k+1}\cup\cdots\cup\mathcal{T}_n$ such that

$$\mathrm{supp}(Q)=\bigcup_{\Phi\in\mathcal{W}}\mathrm{clos}(\Phi^{(k)}).$$

LEMMA 27. *Let $k\leqq i$. Let N be a regular k-submap of $\mathcal{M}$ such that* $\mathrm{int}(N)$ *is connected and let m be maximal such that $\mathcal{U}_m=\mathcal{T}_m\cap\mathrm{Reg}(N)\neq\varnothing$.*

(a) *The ordered m-ranked map $\mathcal{N}=(N,\{\mathcal{U}_1,\cdots\mathcal{U}_m\},<)$ satisfies condition* (SC_k), *and $N^{(h)}$ is a $(k-h)$-submap of $M^{(h)}$ for $h=0,1,\cdots,k$.*

(b) *For any $l,h,h\leqq l\leqq k$, a region $\Phi\in\mathrm{Reg}(N)$ such that $\mathrm{rank}(\Phi)>l$, a vertex $v\in\mathrm{bd}(\Phi^{(l)})$ and a boundary path μ of $\Phi^{(l)}$, we have*

$$\mathrm{LT}_{\mathcal{N}}(v;\Phi^{(h)})=\mathrm{LT}_{\mathcal{M}}(v;\Phi^{(h)}),\qquad \mathrm{RT}_{\mathcal{N}}(v;\Phi^{(h)})=\mathrm{RT}_{\mathcal{M}}(v;\Phi^{(h)}),$$

$$\mathrm{lpr}_{\mathcal{N}}(\mu;\Phi^{(h)}) = \mathrm{lpr}_{\mathcal{M}}(\mu;\Phi^{(h)}), \qquad \mathrm{rpr}_{\mathcal{N}}(\mu;\Phi^{(h)}) = \mathrm{rpr}_{\mathcal{M}}(\mu;\Phi^{(h)}),$$

$$\mathrm{pr}_{\mathcal{N}}(\mu;\Phi^{(h)}) = \mathrm{pr}_{\mathcal{M}}(\mu;\Phi^{(h)}), \qquad \mathrm{S}_{\mathcal{N}}(\mu;\Phi^{(h)}) = \mathrm{S}_{\mathcal{M}}(\mu;\Phi^{(h)}).$$

This lemma immediately follows from Lemma 20 and its Corollary.

5.4. *A technical lemma.*

LEMMA 28. *Let $\mathcal{M} = (M, \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}, <)$ be an ordered n-ranked map satisfying* (SC$_i$) *for some i, $0 \leqq i < n$. Let Φ be a region in M of rank $> i$ and $\Phi^{(i)}$ the corresponding region in $M^{(i)}$. Let μ be a non-trivial p.o.b.p. of $\Phi^{(i)}$. Assume that the following are given*:

(α) *a factorization $\mu = \mu_1\mu_2 \cdots \mu_h$, where each μ_j is non-trivial*;

(β) *a subset S of the set of paths $\{\mu_1, \mu_2, \cdots, \mu_h\}$ such that there is no j, $1 \leqq j < h$, for which both $\mu_j \in S$ and $\mu_{j+1} \in S$*;

(γ) *a factorization* $\mathrm{pr}(\mu;\Phi) = \kappa_1\nu\kappa_2$.

Then there exist factorizations

$$\mu = \theta'\theta_1\theta_2\theta_3\theta'' \tag{13}$$

and

$$\nu = \phi_1\phi_2\phi_3\psi \tag{14}$$

with the following properties:

(a) *If θ_k is non-trivial then $\theta_k = \mu_{j_k}$ for some j_k, $k = 1, 2, 3$.*

(b) *If θ_1 is non-trivial then $\theta_1 \not\in S$ and ϕ_1 is a subpath of* $\mathrm{pr}(\theta_1;\Phi)$. *If θ_1 is trivial then ϕ_1 is trivial.*

(c) *If θ_2 is non-trivial then $\theta_2 \in S$ and for some κ_0*

(α) *$\kappa_0\phi_2$ is a head of* $\mathrm{pr}(\theta_2;\Phi)$;

(β) $\mathrm{lpr}(\theta'\theta_1;\Phi)\kappa_0 = \kappa_1\phi_1$.

If θ_2 is trivial then ϕ_2 is trivial.

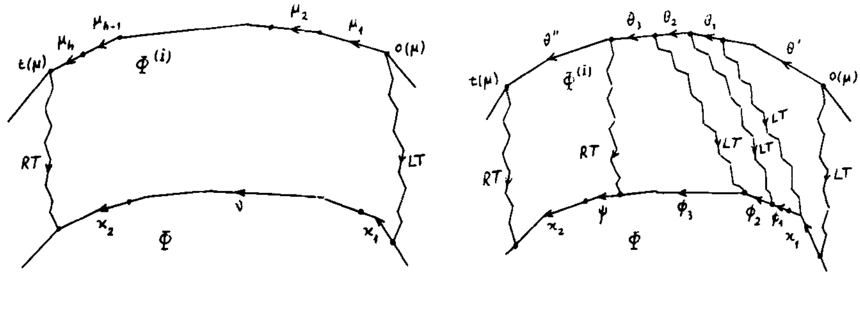


Fig. 59.

(d) *If* θ_3 *is non-trivial, then* $\theta_3 \notin S$, ϕ_3 *is a head of* $\mathrm{pr}(\theta_3;\Phi)$ *and* $\mathrm{lpr}(\theta'\theta_1\theta_2;\Phi)=\kappa_1\phi_1\phi_2$. *If* θ_3 *is trivial then* ϕ_3 *is trivial.*

(e) *If* ψ *is non-trivial, then* θ_3 *and* θ'' *are non-trivial,* $\phi_3=\mathrm{pr}(\theta_3;\phi)$ *and* $\mathrm{rpr}(\theta'';\Phi)=\psi\kappa_2$.

(f) $\theta_1\theta_2$ *is non-trivial.*

(g) *If* κ_1 *is trivial then* θ' *is trivial* (see Fig. 59).

PROOF. If κ_1 is non-trivial, let j be the maximal integer such that $\mathrm{lpr}(\mu_1\cdots\mu_{j-1};\Phi)$ is a head of κ_1; if κ_1 is trivial, let $j=1$.

Let j' be the minimal integer such that $j'\geqq j$ and $\kappa_1\nu$ is a head of $\mathrm{pr}(\mu_1\cdots\mu_{j'};\Phi)$.

For some κ' and κ''

$$\mathrm{pr}(\mu_j\cdots\mu_{j'};\Phi)=\kappa'\nu\kappa'' \tag{15}$$

where

$$\kappa_1=\mathrm{lpr}(\mu_1\cdots\mu_{j-1};\Phi)\kappa',\qquad \kappa_2=\kappa''\mathrm{rpr}(\mu_{j'+1}\cdots\mu_h;\Phi) \tag{16}$$

(see Fig. 60). Define:

$$\theta':=\mu_1\cdots\mu_{j-1} \tag{17}$$

(if $j=1$, this means that $\theta'=\mathrm{o}(\mu)$).

We now consider the different possibilities, specifying in each case the relations that define $\theta_1,\theta_2,\theta_3,\theta'',\phi_1,\phi_2,\phi_3,\psi,\kappa_0$.

Case 1. $j=j'$ and $\mu_j\notin S$.

Take $\theta_1:=\mu_j$, $\theta_2:=\mathrm{t}(\mu_j)$, $\theta_3:=\mathrm{t}(\mu_j)$, $\theta'':=\mu_{j+1}\cdots\mu_h$, $\phi_1:=\nu$, $\phi_2:=\mathrm{t}(\nu)$, $\phi_3:=\mathrm{t}(\nu)$, $\psi:=\mathrm{t}(\nu)$, $\kappa_0:=\mathrm{t}(\nu)$ (see Fig. 61).

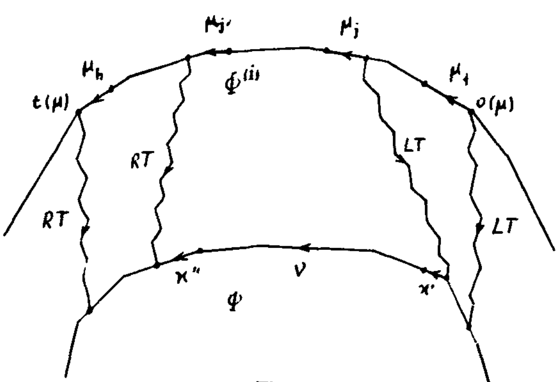


Fig. 60.

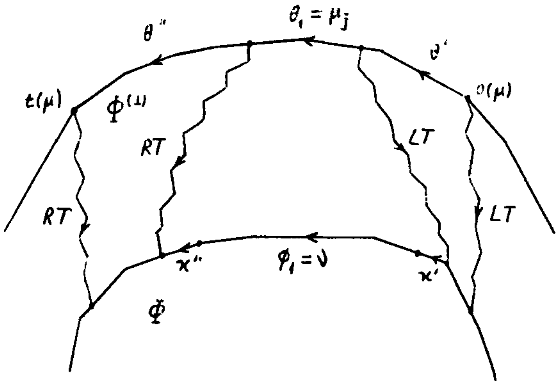


Fig. 61.

Case 2. $j = j'$ and $\mu_j \in S$.

Take $\theta_1 := \mathrm{o}(\mu_j)$, $\theta_2 := \mu_j$, $\theta_3 := \mathrm{t}(\mu_j)$, $\theta' := \mu_{j+1} \cdots \mu_h$, $\phi_1 := \mathrm{o}(\nu)$, $\phi_2 := \nu$, $\phi_3 := \mathrm{t}(\nu)$, $\psi := \mathrm{t}(\nu)$, $\kappa_0 := \kappa'$ (see Fig. 62).

Case 3. $j' = j + 1$, $\mu_j \notin S$ and $\mu_{j+1} \in S$.

Take $\theta_1 := \mu_j$, $\theta_2 := \mu_{j+1}$, $\theta_3 := \mathrm{t}(\mu_{j+1})$, $\theta'' := \mu_{j+2} \cdots \mu_h$. Since j is maximal, κ' is a head of $\mathrm{lpr}(\mu_j; \Phi)$, and since j' is minimal, κ'' is a tail of $\mathrm{pr}(\mu_{j+1}; \Phi)$. Hence there exists a factorization $\nu = \phi_1 \phi_2$ such that

$$\mathrm{lpr}(\theta_1; \Phi) = \kappa' \phi_1, \qquad \mathrm{pr}(\theta_2; \Phi) = \phi_2 \kappa''.$$

Take $\phi_3 := \mathrm{t}(\nu)$, $\psi := \mathrm{t}(\nu)$, $\kappa_0 := \mathrm{o}(\phi_2)$ (see Fig. 63).

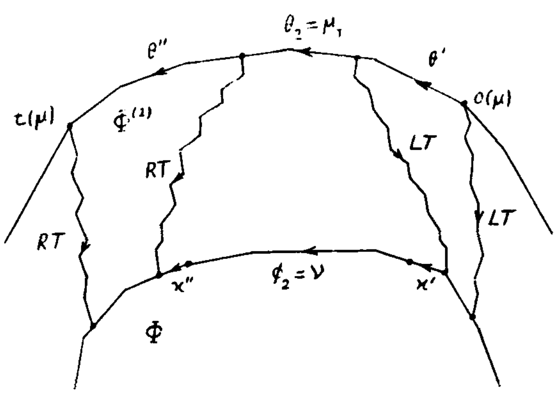


Fig. 62.

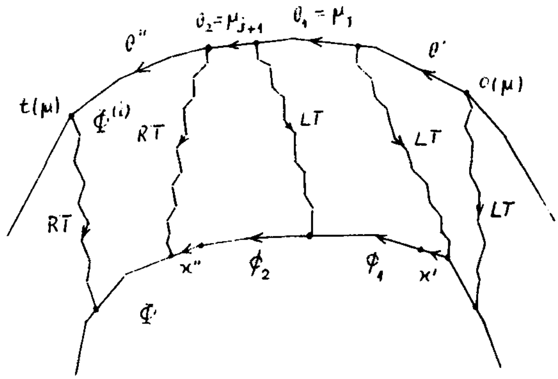


Fig. 63.

Case 4. $j' = j+1$, $\mu_j \notin S$ and $\mu_{j+1} \notin S$.

Take $\theta_1 := \mu_j$, $\theta_2 := \mathrm{t}(\mu_j)$, $\theta_3 := \mu_{j+1}$, $\theta'' := \mu_{j+2} \cdots \mu_h$. There is a factorization $\nu = \phi_1 \phi_3$ such that

$$\mathrm{lpr}(\theta_1; \Phi) = \kappa' \phi_1, \qquad \mathrm{pr}(\theta_3; \Phi) = \phi_3 \kappa''.$$

Take also $\phi_2 := \mathrm{t}(\phi_1)$, $\psi := \mathrm{t}(\nu)$, $\kappa_0 := \mathrm{o}(\phi_3) = \mathrm{t}(\phi_1)$ (see Fig. 64).

Case 5. $j' = j+1$, $\mu_j \in S$ and $\mu_{j+1} \notin S$.

Take $\theta_1 := \mathrm{o}(\mu_j)$, $\theta_2 := \mu_j$, $\theta_3 := \mu_{j+1}$, $\theta'' := \mu_{j+2} \cdots \mu_h$. There is a factorization $\nu = \phi_2 \phi_3$ such that

$$\mathrm{lpr}(\theta_2; \Phi) = \kappa' \phi_2, \qquad \mathrm{pr}(\theta_3; \Phi) = \phi_3 \kappa''.$$

Take also $\phi_1 := \mathrm{o}(\nu)$, $\psi := \mathrm{t}(\nu)$, $\kappa_0 := \kappa'$ (see Fig. 65).

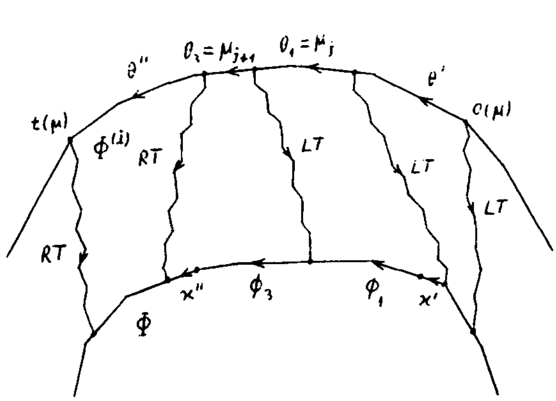


Fig. 64.

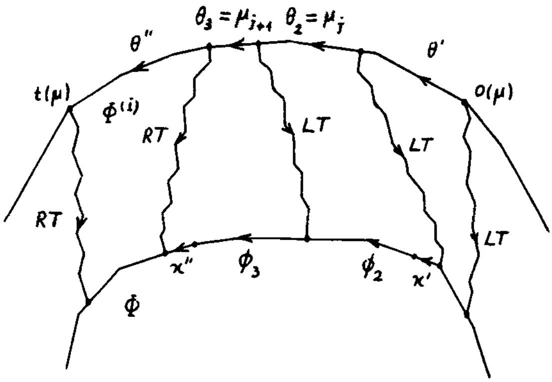


Fig. 65.

Case 6. $j' = j + 2$, $\mu_j \notin S$, $\mu_{j+1} \in S$, $\mu_{j+2} \notin S$.

Take $\theta_1 := \mu_j$, $\theta_2 := \mu_{j+1}$, $\theta_3 := \mu_{j+2}$, $\theta'' := \mu_{j+3} \cdots \mu_h$. There is a factorization $\nu = \phi_1\phi_2\phi_3$ such that

$$\mathrm{lpr}(\theta_1; \Phi) = \kappa'\phi_1, \quad \mathrm{lpr}(\theta_2; \Phi) = \phi_2, \quad \mathrm{pr}(\theta_3; \Phi) = \phi_3\kappa''.$$

Take also $\psi := \mathrm{t}(\nu)$, $\kappa_0 := \mathrm{o}(\phi_2)$ (see Fig. 66).

Case 7. $j' \geqq j + 2$, $\mu_j \notin S$ and $\mu_{j+1} \notin S$.

Take $\theta_1 := \mu_j$, $\theta_2 := \mathrm{t}(\mu_j)$, $\theta_3 := \mu_{j+1}$, $\theta'' := \mu_{j+2} \cdots \mu_h$. There is a factorization $\nu = \phi_1\phi_3\psi$ such that

$$\mathrm{lpr}(\theta_1; \Phi) = \kappa'\phi_1, \quad \mathrm{pr}(\theta_3; \Phi) = \phi_3, \quad \mathrm{rpr}(\mu_{j+2} \cdots \mu_{j'}; \Phi) = \psi\kappa''.$$

Take also $\phi_2 := \mathrm{t}(\phi_1)$, $\kappa_0 := \mathrm{t}(\phi_1)$ (see Fig. 67).

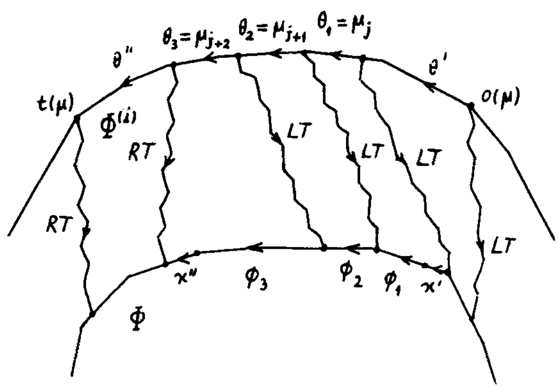


Fig. 66.

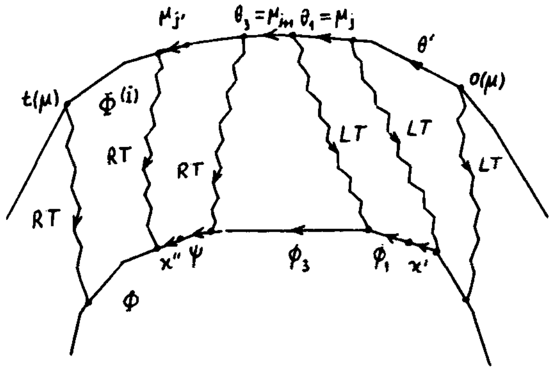


Fig. 67.

Case 8. $j' \geqq j+2$, $\mu_j \in S$ and $\mu_{j+1} \notin S$.

Take $\theta_1 := o(\mu_j)$, $\theta_2 := \mu_j$, $\theta_3 := \mu_{j+1}$, $\theta'' := \mu_{j+2} \cdots \mu_h$. There is a factorization $\nu = \phi_2 \phi_3 \psi$ such that

$$\mathrm{lpr}(\theta_2; \Phi) = \kappa' \phi_2, \quad \mathrm{pr}(\theta_3; \Phi) = \phi_3, \quad \mathrm{rpr}(\mu_{j+2} \cdots \mu_{j'}; \Phi) = \psi \kappa''.$$

Take also $\phi_1 := o(\nu)$, $\kappa_0 := \kappa'$ (see Fig. 68).

Case 9. $j' > j+2$, $\mu_j \notin S$, $\mu_{j+1} \in S$ and $\mu_{j+2} \notin S$.

Take $\theta_1 := \mu_j$, $\theta_2 := \mu_{j+1}$, $\theta_3 := \mu_{j+2}$, $\theta'' := \mu_{j+3} \cdots \mu_h$. There is a factorization $\nu = \phi_1 \phi_2 \phi_3 \psi$ such that

$$\mathrm{lpr}(\theta_1; \Phi) = \kappa' \phi_1, \quad \mathrm{lpr}(\theta_2; \Phi) = \phi_2, \quad \mathrm{pr}(\theta_3; \Phi) = \phi_3, \quad \mathrm{rpr}(\mu_{j+3} \cdots \mu_{j'}; \Phi) = \psi \kappa''.$$

Take $\kappa_0 := o(\phi_2)$ (see Fig. 69).

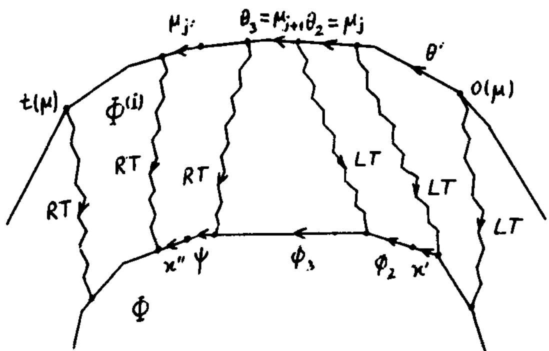


Fig. 68.

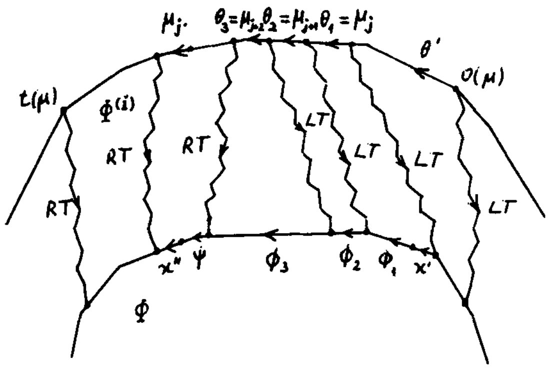


Fig. 69.

It is easy to check that these 9 cases exhaust all the possibilities. In each case we have factorizations $\mu = \theta'\theta_1\theta_2\theta_3\theta''$ and $\nu = \phi_1\phi_2\phi_3\psi$ that satisfy conditions (a), (b), (c), (d), (e), (f), (g), and so the lemma is proved.

§6. Paths on the common boundary of regions in $M^{(i)}$

The following theorem is the central result of the theory.

THEOREM 4. *Let $\mathcal{M} = (M, \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}, <)$ be an ordered n-ranked map satisfying condition* (S_0) (see 2.4). *Let i be some integer, $0 \leqq i < n$. Assume that $\mathcal{M}$ satisfies* (SC_i).

Let Φ and Ψ be regions in M, of ranks $r > i$ and $s > i$, respectively. Since $\mathcal{M}$ satisfies (SC_i), *we can speak of the ordered $(n-i)$-ranked map $\mathcal{M}^{(i)} = (M^{(i)}, \{\mathcal{T}^{(i)}_{i+1}, \cdots, \mathcal{T}^{(i)}_n\}, <)$. Consider the regions $\Phi^{(i)}$ and $\Psi^{(i)}$ in $M^{(i)}$ corresponding to Φ and Ψ. Let μ be a non-trivial p.o.b.p. of $\Phi^{(i)}$ which is also a n.o.b.p. of $\Psi^{(i)}$. Then:*

$$\text{(1)} \qquad \mathrm{pr}(\mu;\Phi) \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j + e_s\right)$$

(see Definition 9) (see Fig. 70).

Moreover, let τ be a subpath of $\mathrm{pr}(\mu;\Phi)$, *i.e., for some ω', ω'',*

$$\text{(2)} \qquad \mathrm{pr}(\mu;\Phi) = \omega'\tau\omega''.$$

Then either

$$\text{(3)} \qquad \tau \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j\right)$$

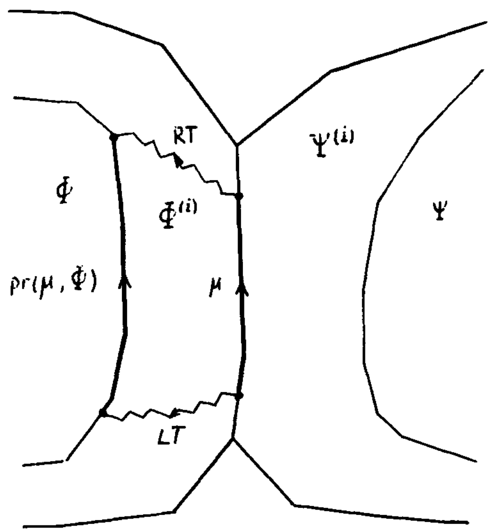


Fig. 70.

or there exists a factorization

$$\tau = \tau_1 \theta \tau_2 \tag{4}$$

such that

$$\tau_1, \tau_2 \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j\right) \tag{5}$$

and

$$\theta \in \mathscr{I}(\Phi; e_s) = \mathscr{P}(\Phi; s) \tag{6}$$

(see Fig. 71).

More precisely, there exist two simple *paths* η, η' *and a boundary path* ξ *of* Ψ *such that*

$$\eta, \eta' \in \mathrm{Br}(i), \quad \theta \underset{i}{\sim} \eta \xi \eta'^{-1} \tag{7}$$

(see Definitions 8, 9) *and* η, η' *have the following additional properties*:

(A) *There exists a factorization* $\eta = \eta_1 \eta_2$ *such that*

(α) $t(\eta_1) = o(\eta_2)$ *is a vertex on* μ, η_1 *is a path in* $S(\mu; \Phi)$ *and* η_2 *is a path in* $S(\mu; \Psi)$;

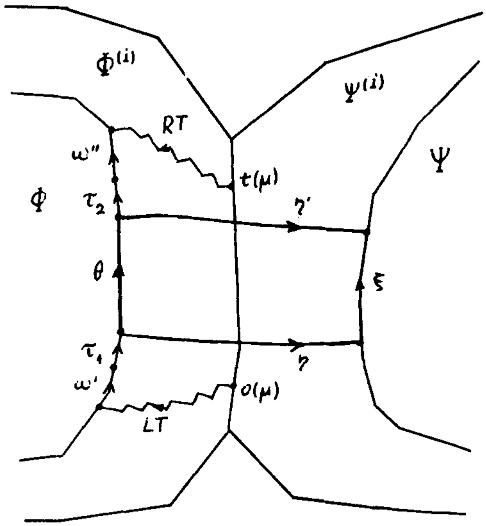


Fig. 71.

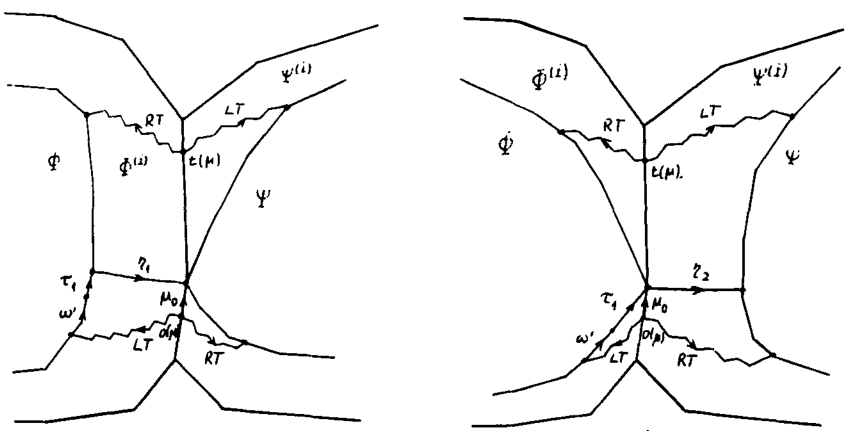
(β) *denoting by* μ_0 *the head of* μ *such that* $\mathrm{t}(\mu_0)=\mathrm{t}(\eta_1)$, *we have*

$$\omega'\tau_1 \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu_0\eta_1^{-1};$$

(γ) *if* $\Phi<\Psi$, *then* η_2 *is trivial*; *if* $\Psi<\Phi$, *then* η_1 *is trivial* (see Fig. 72).

(B) *The vertex* $\mathrm{t}(\eta)$ *is on the path* $\mathrm{pr}(\mu;\Psi)$. *If* ω' *is trivial, then, denoting by* τ_3 *the (minimal) head of* $\mathrm{pr}(\mu;\Psi)$ *such that* $\mathrm{t}(\eta)=\mathrm{t}(\tau_3)$, *we have*

(α) $$\tau_3\in\mathscr{H}\left(\Psi;\ \sum_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j\right);$$



Φ < Ψ Ψ < Φ

Fig. 72.

(β) $\tau_3 \sim_i \mathrm{RT}(\mathrm{o}(\mu);\Psi)^{-1}\mu_0\eta_2$ (see Figs. 72 and 73).

Similarly:

(A′) *There exists a factorization* $\eta' = \eta_1'\eta_2'$ *such that*

(α) $\mathrm{t}(\eta_1') = \mathrm{o}(\eta_2')$ *is a vertex on* μ, η_1' *is a path in* $\mathrm{S}(\mu;\Phi)$ *and* η_2' *is a path in* $\mathrm{S}(\mu;\Psi)$;

(β) *denoting by* μ_0' *the tail of* μ *such that* $\mathrm{o}(\mu_0') = \mathrm{t}(\eta_1')$, *we have*

$$\tau_2\omega'' \underset{i}{\sim} \eta_1'\mu_0'\mathrm{RT}(\mathrm{t}(\mu);\Phi);$$

(γ) *if* $\Phi < \Psi$, *then* η_2' *is trivial*; *if* $\Psi < \Phi$ *then* η_1' *is trivial.*

(B′) *The vertex* $\mathrm{t}(\eta')$ *is on the path* $\mathrm{pr}(\mu;\Psi)$. *If* ω'' *is trivial then, denoting by* τ_4 *the (minimal) tail of* $\mathrm{pr}(\mu;\Psi)$ *such that* $\mathrm{t}(\eta') = \mathrm{o}(\tau_4)$, *we have*

(α) $\tau_4 \in \mathscr{H}(\Psi;\Sigma_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j)$;

(β) $\tau_4 \sim_i \eta_2'^{-1}\mu_0'\mathrm{LT}(\mathrm{t}(\mu);\Psi)$.

COROLLARY 1. *Let* $\mathcal{M}$ *be an ordered n-ranked map satisfying condition* (S_0) *and condition* (SC_i) *for some i*, $0 \leqq i < n$. *Let* $\mathcal{M}^{(i)} = (M^{(i)}, \{\mathcal{T}_{i+1}^{(i)}, \cdots, \mathcal{T}_n^{(i)}\}, <)$ *be the ordered* $(n-i)$*-ranked map defined in* 5.1. *Recall that* $\mathcal{T}_{i+k}^{(i)}$ *is the set of regions of rank k of* $\mathcal{M}^{(i)}$.

Let $\Phi \in \mathcal{T}_r$, $r > i$, *and let* ν *be a non-trivial boundary path of* $\Phi^{(i)}$. *If* $\nu \in \Phi^{(i)}(\Sigma_{k\geq 1} c_k e_k)$ *in* $\mathcal{M}^{(i)}$ (see Definition 30), *then*

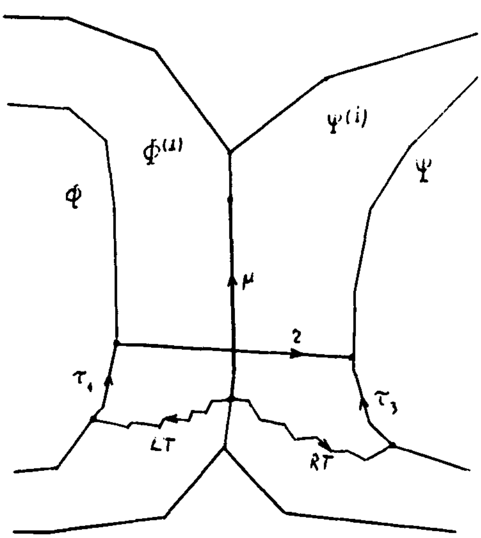


Fig. 73.

$$\operatorname{pr}(\nu;\Phi) \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} c \cdot 13^{i+1-j} e_j + \sum_{k \geqq 1} c_k e_{i+k}\right)$$

where $c = \Sigma_{k \geqq 1} c_k$.

PROOF. By Lemma 1(a), Lemma 15(g) and Lemma 26(a), we may assume without loss of generality that ν is a p.o.b.p. of $\Phi^{(i)}$. By Definition 9, we can write $\nu = \nu_1 \nu_2 \cdots \nu_m$, where each ν_h is a n.o.b.p. of some region $\Psi_h^{(i)} \in \mathscr{T}_{q(h)}^{(i)}$, $q(h) > i$, and for any $k \geqq 1$,

$$\operatorname{card}\{h \mid 1 \leqq h \leqq m, \operatorname{rank}_{\mathscr{M}^{(i)}}(\Psi_h^{(i)}) = k\} = \operatorname{card}\{h \mid 1 \leqq h \leqq m, q(h) = i + h\} \leqq c_k.$$

By Lemma 7(d) and Lemma 26(b),

$$\operatorname{pr}(\nu;\Phi) = \operatorname{pr}(\nu_1 \nu_2 \cdots \nu_m;\Phi) = \sigma_1 \sigma_2 \cdots \sigma_m$$

where each σ_h (if non-trivial) is a subpath of $\operatorname{pr}(\nu_j;\Phi)$. By (1),

$$\operatorname{pr}(\nu_h;\Phi) \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j + e_{q(h)}\right),$$

and hence, by Lemma 1(b), also

$$\sigma_h \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j + e_{q(h)}\right).$$

Since $m \leqq c = \Sigma_{k>1} c_k$ and $\Sigma_{h=1}^{m} e_{q(h)} \leqq \Sigma_{k \geqq 1} c_k e_{i+k}$, we obtain

$$\operatorname{pr}(\nu;\Phi) = \sigma_1 \sigma_2 \cdots \sigma_m \in \mathscr{H}\left(\Phi; \sum_{h=1}^{m} \left(\sum_{j=1}^{i} 13^{i+1-j} e_j + e_{q(h)}\right)\right)$$

$$\subseteq \mathscr{H}\left(\Phi; \sum_{j=1}^{i} m \cdot 13^{i+1-j} e_j + \sum_{h=1}^{m} e_{q(h)}\right) \subseteq \mathscr{H}\left(\Phi; \sum_{j=1}^{i} c \cdot 13^{i+1-j} e_j + \sum_{k \geqq 1} c_k e_{i+k}\right).$$

The corollary is proved.

COROLLARY 2. *Let $\mathscr{M}$ be an ordered n-ranked map satisfying condition* (S_0) *and condition* (SC_i) *for some i, $0 \leqq i < n$. Then $\tilde{\mathscr{M}}^{(i)}$ satisfies conditions* D(8) *and* D(6; 1).

PROOF. Suppose that there is a region $\Phi^{(i)} \in \mathscr{T}_{i+1}^{(i)}$ with a boundary cycle ν such that $\nu \in \Phi^{(i)}(6e_i + e_2)$ in $\tilde{\mathscr{M}}^{(i)}$. Then, by 5.1, for some $k \geqq 2$, $\nu \in \Phi^{(i)}(6e_i + e_k)$ in $\mathscr{M}^{(i)}$. By Corollary 1,

$$\operatorname{pr}(\nu;\Phi) \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} 7 \cdot 13^{i+1-j} e_j + 6e_{i+1} + e_{i+k}\right).$$

By Lemma 7(f) and Lemma 26(b), there is a head σ of $\mathrm{pr}(\nu;\Phi)$ which is a boundary cycle of Φ, and then

$$\sigma \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} 7\cdot 13^{i+1-j}e_j + 6e_{i+1} + e_{i+k}\right) \subseteq \mathscr{I}\left(\Phi; \sum_{j=1}^{i} 7\cdot 13^{i+1-j}e_j + 6e_{i+1} + e_{i+k}\right).$$

This contradicts (S_0) since $\Phi \in \mathscr{T}_{i+1}$. Therefore there is no such $\Phi^{(i)}$ in $\tilde{\mathscr{M}}^{(i)}$, and so $\tilde{\mathscr{M}}^{(i)}$ satisfies D(6; 1).

The other assertion can be proved in similar fashion.

The corollary is proved.

Proof of Theorem 4. We proceed by induction on i.

Consider the case $i = 0$. Then $\Phi = \Phi^{(0)}$, $\Psi = \Psi^{(0)}$, μ is on the common boundary of Φ and Ψ, and therefore $\mathrm{pr}(\mu;\Phi) = \mu = \mathrm{pr}(\mu;\Psi)$. Since $\mathscr{M}$ satisfies (SC_0), $\mathrm{clos}(\Pi)$ is simply-connected for each region $\Pi \in \mathrm{Reg}(M)$. The path μ is non-trivial, Φ is to the left of μ and Ψ is to the right of μ (see Fig. 74), and so $\Phi \neq \Psi$. By assumption, $\Phi \in \mathscr{T}_r$ and $\Psi \in \mathscr{T}_s$. Hence, by Definition 9, each subpath of μ belongs to $\mathscr{I}(\Phi; e_s)$. Then, by Definition 9,

$$\mathrm{pr}(\mu;\Phi) = \mu \in \mathscr{H}(\Phi; e_s).$$

Let $\mathrm{pr}(\mu;\Phi) = \omega'\tau\omega''$. Take $\eta := \mathrm{o}(\tau)$, $\eta' := \mathrm{t}(\tau)$, $\tau_1 := \mathrm{o}(\tau)$, $\theta := \tau$, $\tau_2 := \mathrm{t}(\tau)$, $\xi := \tau$. By Definition 9, η and η' belong to Br(0). Conditions (4), (5), (6), (7), (A), (B), (A′), (B′) are obviously satisfied (see Fig. 75).

Now let $i > 0$. By 5.1, condition (SC_i) implies (SC_l) for any $l < i$. Therefore, by induction hypothesis, we have:

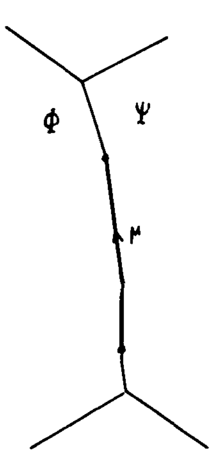


Fig. 74.

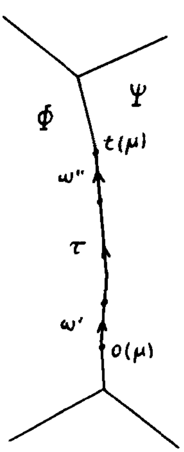


Fig. 75.

1°. All the assertions of Theorem 4, Corollary 1 and Corollary 2 hold whenever i is replaced by any $l < i$.

In particular, for any $j < i$, we have:

2°. The ordered 2-ranked map

$$\bar{\mathcal{M}}^{j} = (M^{(j)}, \{\mathcal{T}^{(j)}_{j+1}, \mathcal{T}^{(j)}_{j+2} \cup \mathcal{T}^{(j)}_{j+3} \cup \cdots \cup \mathcal{T}^{(j)}_{n}\}, <)$$

satisfies conditions D(8) and D(6; 1).

Since $\mathcal{M}$ satisfies (SC_{j+1}) for any $j < i$, it follows from 5.1 that:

3°. $\bar{\mathcal{M}}^{(j)}$ satisfies condition (SC).

4°. Let $0 \leqq l \leqq i$. Let Γ be a region in M of rank $> l$. Let v be a vertex on the boundary of $\Gamma^{(l)}$. Then

$$\operatorname{pr}(v; \Gamma) \in \mathcal{H}\left(\Gamma; \sum_{j=1}^{l} 2 \cdot 13^{l-j} e_j\right).$$

We proceed by induction on l. If $l = 0$, there is nothing to prove. Let $l > 0$. If $\operatorname{pr}(v; \Gamma^{(l-1)})$ is a single vertex, then by the induction hypothesis,

$$\operatorname{pr}(v; \Gamma) = \operatorname{pr}(\operatorname{pr}(v; \Gamma^{(l-1)}); \Gamma) \in \mathcal{H}\left(\Gamma; \sum_{j=1}^{l-1} 2 \cdot 13^{l-1-j} e_j\right) \subseteq \mathcal{H}\left(\Gamma; \sum_{j=1}^{l} 2 \cdot 13^{l-j} e_j\right).$$

Assume, then, that $\operatorname{pr}(v; \Gamma^{(l-1)})$ is a non-trivial path. In view of 2° and 3°, Lemma 24 gives

$$\operatorname{pr}(v, \Gamma^{(l-1)}) \in \Gamma^{(l-1)}(2e_1)$$

in $\bar{\mathcal{M}}^{(l-1)}$, hence in $\mathcal{M}^{(l-1)}$. Hence, by 1° and Corollary 1,

$$\operatorname{pr}(v; \Gamma) = \operatorname{pr}(\operatorname{pr}(v; \Gamma^{(l-1)}); \Gamma) \in \mathcal{H}\left(\Gamma; \sum_{j=1}^{l-1} 2 \cdot 13^{l-j} e_j + 2e_l\right) = \mathcal{H}\left(\Gamma; \sum_{j=1}^{l} 2 \cdot 13^{l-j} e_j\right).$$

5°. Let $0 \leqq l \leqq i$ and let Γ be a region in M of rank $> l$. Then $\operatorname{clos}(\Gamma^{(l)})$ is simply-connected.

Indeed, if $l = 0$, then $\Gamma^{(l)} = \Gamma$, and then $\operatorname{clos}(\Gamma) = \operatorname{clos}(\Gamma^{(l)})$ is simply-connected since $\mathcal{M}$ satisfies (SC_0). If $l \geqq 1$, then $\Gamma^{(l)}$ is a region in $M^{(l)}$, the derived map of the ordered 2-ranked map $\bar{\mathcal{M}}^{(l-1)}$. By 3°, $\bar{\mathcal{M}}^{(l-1)}$ satisfies (SC) and therefore $\operatorname{clos}(\Gamma^{(l)})$ is simply-connected.

By assumption, μ is a non-trivial path which is a p.o.b.p. of $\Phi^{(i)}$ and a n.o.b.p. of $\Psi^{(i)}$. In view of 5°, we obtain:

6°. $\Phi \neq \Psi$ and μ does not contain a boundary cycle of $\Phi^{(i)}$. In particular, μ is simple.

Our next goal is to prove the following statement:

(C) Either $\tau \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j} e_j)$, or there exists a simple path $\eta \in \mathrm{Br}(i)$ connecting a vertex of τ to a vertex of $\mathrm{pr}(\mu; \Psi)$, having properties (A), (B) and such that, if τ_1 is the (minimal) head of τ satisfying $\mathrm{t}(\tau_1) = \mathrm{o}(\eta)$, then $\tau_1 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j)$.

Applying Proposition 2 with $\mathcal{M}, \Phi, \Psi, \Phi', \Psi'$ replaced by $\tilde{\mathcal{M}}^{(i-1)}$, $\Phi^{(i-1)}$, $\Psi^{(i-1)}$, $\Phi^{(i)}$, $\Psi^{(i)}$, we obtain a factorization

$$\mu = \mu' \mu'' \mu''' \tag{8}$$

and, if μ'' is non-trivial, a factorization

$$\mu'' = \mu_1 \mu_2 \cdots \mu_h \tag{9}$$

with the properties described in Proposition 2.

By Lemma 7(d) and Lemma 26(b), there is a factorization

$$\tau = \tau' \tau'' \tau''' \tag{10}$$

with the following properties:

7°. If τ' (τ'', τ''') is non-trivial, it is a subpath of $\mathrm{pr}(\mu'; \Phi)$ (of $\mathrm{pr}(\mu''; \Phi)$, of $\mathrm{pr}(\mu'''; \Phi)$). Moreover, there are paths κ_1, κ_2 such that

(α) $\mathrm{pr}(\mu''; \Phi) = \kappa_1 \tau'' \kappa_2$;

(β) $\mathrm{lpr}(\mu'; \Phi) \kappa_1 = \omega' \tau'$;

(γ) $\kappa_2 \,\mathrm{rpr}(\mu'''; \Phi) = \tau''' \omega''$ (see Fig. 76).

8°. If μ'' is trivial then τ'' is trivial.

9°. If ω' is trivial and τ'' is non-trivial, then κ_1 is trivial.

By Proposition 2(c), $\mathrm{pr}(\mu'; \Phi^{(i-1)}) \in \Phi^{(i-1)}(2e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$, hence in $\mathcal{M}^{(i-1)}$, then, by the induction hypothesis and Corollary 1,

$$\mathrm{pr}(\mu'; \Phi) = \mathrm{pr}(\mathrm{pr}(\mu'; \Phi^{(i-1)}); \Phi) \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} 2 \cdot 13^{i-j} e_j\right).$$

Similarly,

$$\mathrm{pr}(\mu'''; \Phi) \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} 2 \cdot 13^{i-j} e_j\right).$$

In view of 7°, we have

10°. $\tau' \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} 2 \cdot 13^{i-j} e_j)$ and $\tau''' \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} 2 \cdot 13^{i-j} e_j)$.

Similarly, we obtain from Proposition 2(d)

11°. $\mathrm{pr}(\mu'; \Psi) \in \mathscr{H}(\Psi; \Sigma_{j=1}^{i} 4 \cdot 13^{i-j} e_j)$.

Using Proposition 2(h), (i) and (i′) we have:

12°. If μ_1 is not on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$, then $\mathrm{pr}(\mu' \mu_1; \Psi) \in \mathscr{H}(\Psi; \Sigma_{j=1}^{i} 5 \cdot 13^{i-j} e_j)$.

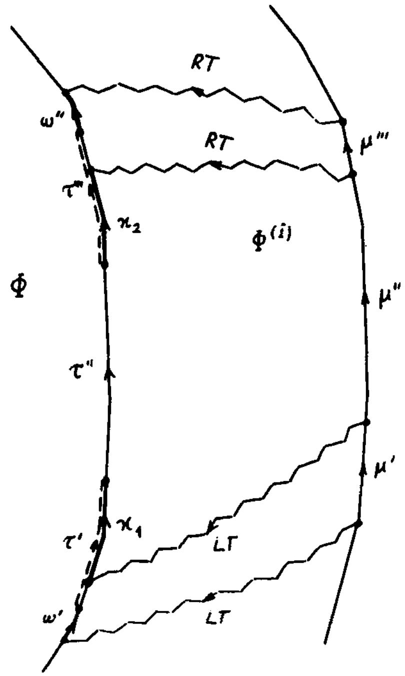


Fig. 76.

Comparing (10) and 10° we obtain

13°. If τ'' is trivial then $\tau = \tau'\tau''' \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 4\cdot 13^{i-j}e_j) \subseteq \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j}e_j)$.

In what follows, we assume that τ'' is non-trivial. Then, by 8°, μ'' is also non-trivial and there is a factorization (9).

Let S be the subset of $\{\mu_1, \mu_2, \cdots, \mu_h\}$ defined as follows:

(11) $$S := \{\mu_j \mid \mu_j \text{ is on the common boundary of } \Phi^{(i-1)} \text{ and } \Psi^{(i-1)}\}.$$

If $\Phi < \Psi$ then, by Proposition 2(g), (9) is the left-hand-side factorization of μ'' in $M^{(i-1)}$. By 5°, clos($\Phi^{(i-1)}$) is simply-connected and therefore μ_j and μ_{j+1} cannot both be on the boundary of $\Phi^{(i-1)}$. Hence either $\mu_j \notin S$ or $\mu_{j+1} \notin S$. If $\Psi < \Phi$, then we reach the same conclusion using Proposition 2(g′).

Now apply Lemma 28 with μ, ν replaced by μ'', τ''. There result factorizations

(12) $$\mu'' = \theta'\theta_1\theta_2\theta_3\theta''$$

and

(13) $$\tau'' = \phi_1\phi_2\phi_3\psi$$

with the properties described in Lemma 28.

We have

14°. $\phi_1 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} 13^{i-j} e_j)$.

This is clear if ϕ_1 is trivial. If ϕ_1 is non-trivial, then, in view of Lemma 28(b), θ_1 is non-trivial. Then by Lemma 28(a), (b), $\theta_1 = \mu_{j_1} \not\in S$. Then, by (11) and Proposition 2(h), (h′), we obtain

$$\mathrm{pr}(\mu_{j_1}; \Phi^{(i-j)}) \in \Phi^{(i-1)}(e_1)$$

in $\tilde{\mathscr{M}}^{(i-1)}$, hence in $\mathscr{M}^{(i-1)}$. Then, by the induction hypothesis and Corollary 1, we have

$$\mathrm{pr}(\theta_1; \Phi) = \mathrm{pr}(\mu_{j_1}; \Phi) = \mathrm{pr}(\mathrm{pr}(\mu_{j_1}; \Phi^{(i-1)}); \Phi) \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i} 13^{i-j} e_j\right) \tag{14}$$

and then, by Lemma 28(b), also $\phi_1 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} 13^{i-j} e_j)$, as required.

Similarly, using Lemma 28(d), we obtain:

15°. $\phi_3 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} 13^{i-j} e_j)$.

We have the following possibilities:

(1) $\phi_2 \not\in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$;

(2) $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$ and ψ is trivial;

(3) $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$, ψ is non-trivial, ω' is trivial and $\mathrm{rpr}(\theta_2; \Psi) \not\in \mathscr{H}(\Psi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$;

(4) $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$, ψ is non-trivial and either ω' is non-trivial or $\mathrm{rpr}(\theta_2; \Psi) \in \mathscr{H}(\Psi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$.

We consider each of these cases separately.

Case 1. $\phi_2 \not\in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$.

In this case ϕ_2 is non-trivial. Hence, by Lemma 28(c), θ_2 is non-trivial, and then $\theta_2 = \mu_{j_2} \in S$. By (11), μ_{j_2} is on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$.

In view of Lemma 28(c) (α) we have paths κ_0, κ_0' such that $\mathrm{pr}(\mu_{j_2}; \Phi) = \kappa_0 \phi_2 \kappa_0'$ (see Fig. 77).

Apply the induction hypothesis with $i, \mu, \omega', \tau, \omega''$ replaced by $i-1, \mu_{j_2} = \theta_2, \kappa_0, \phi_2, \kappa_0'$.

Since $\phi_2 \not\in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$, it follows that there is a simple path $\eta \in \mathrm{Br}(i-1)$ connecting a vertex of ϕ_2 to a vertex of $\mathrm{pr}(\mu_{j_2}; \Psi)$ and having the following properties:

16°. Let ν_1 be the (minimal) head of ϕ_2 such that $\mathrm{t}(\nu_1) = \mathrm{o}(\eta)$. Then $\nu_1 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} \frac{1}{2} \cdot 13^{i-j} e_j)$.

17°. There is a factorization $\eta = \eta_1 \eta_2$ such that (see Fig. 78)

(α) $\mathrm{t}(\eta_1) = \mathrm{o}(\eta_2)$ is a vertex on μ_{j_2}, η_1 is a path in $\mathrm{S}(\mu_{j_2}; \Phi)$ and η_2 is a path in $\mathrm{S}(\mu_{j_2}; \Psi)$;

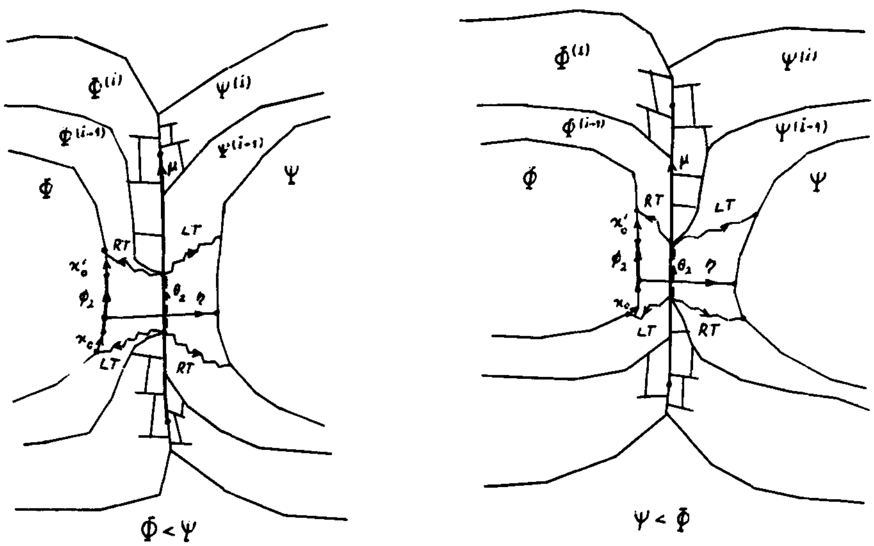


Fig. 77.

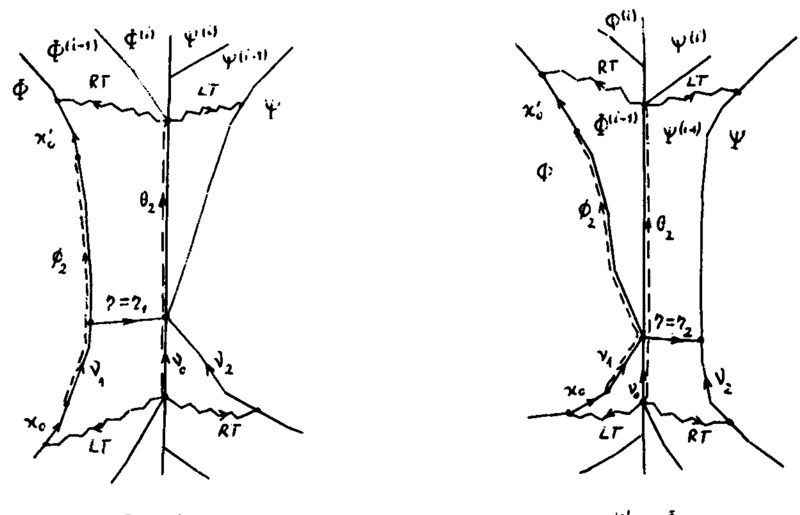


Fig. 78.

(β) denoting by ν_0 the head of $\mu_{\hbar}$ such that $t(\nu_0) = t(\eta_1)$, we have

$$\kappa_0 \nu_1 \underset{i-1}{\simeq} \mathrm{LT}(o(\mu_{\hbar}); \Phi)^{-1} \nu_0 \eta_1^{-1};$$

(γ) if $\Phi < \Psi$, then η_2 is trivial; if $\Psi < \Phi$, then η_1 is trivial.

18°. $t(\eta)$ is a vertex on $\mathrm{pr}(\mu_{\hbar}; \Psi)$. If κ_0 is trivial then, letting ν_2 denote the (minimal) head of $\mathrm{pr}(\mu_{\hbar}; \Psi)$ such that $t(\eta) = t(\nu_2)$, we have

$$\nu_2 \in \mathscr{H}\left(\Psi; \sum_{j=1}^{i-1} \frac{1}{2} \cdot 13^{i-j} e_j\right) \quad \text{and} \quad \nu_2 \underset{i-1}{\simeq} \mathrm{RT}(\mathrm{o}(\mu_{j_2}); \Psi)^{-1} \nu_0 \eta_2.$$

We can now prove (C).

Since by Lemma 1(c) $\mathrm{Br}(i-1) \subseteq \mathrm{Br}(i)$, η is a simple path belonging to $\mathrm{Br}(i)$ and connecting a vertex of ϕ_2, hence of τ (cf. (10) and (13)), to a vertex of $\mathrm{pr}(\mu_{j_2}; \Psi)$, hence of $\mathrm{pr}(\mu; \Psi)$ (cf. (8) and (9)).

Define $\tau_1 := \tau' \phi_1 \nu_1$. Then, by (10), (13) and 16°, τ_1 is a head of τ such that $\mathrm{t}(\tau_1) = \mathrm{o}(\eta)$. By 10°, 14° and 16°,

$$\tau_1 = \tau' \phi_1 \nu_1 \in \mathscr{H}\left(\Phi; \sum_{j=1}^{i-1} 3 \cdot \frac{1}{2} 13^{i-j} e_j + 3e_i\right) \subseteq \mathscr{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j\right).$$

Since $\mathscr{M}$ satisfies (S_0), τ_1 cannot contain a boundary cycle of Φ and therefore τ_1 is the minimal head of τ such that $\mathrm{t}(\tau_1) = \mathrm{o}(\eta)$.

By Definitions 20, 27 and 32, the map $\mathrm{S}(\mu_{j_2}; \Phi)$ is a submap of $\mathrm{S}(\mu; \Phi)$ and $\mathrm{S}(\mu_{j_2}; \Psi)$ is a submap of $\mathrm{S}(\mu; \Psi)$ because by (8) and (9) μ_{j_2} is a subpath of μ. Therefore (A(α)) follows from 17°(α). (A(γ)) follows from 17°(γ). As we know, $\theta_2 = \mu_{j_2}$. Therefore, by (8), (12) and 17°(β), $\mu' \theta' \theta_1 \nu_0$ is a head of μ such that $\mathrm{t}(\mu' \theta' \theta_1 \nu_0) = \mathrm{t}(\nu_0) = \mathrm{t}(\eta_1)$ and hence

$$\mu_0 = \mu' \theta' \theta_1 \nu_0. \tag{15}$$

By 7°(β), Lemma 15(c) and Lemma 26(a),

$$\omega' \tau' \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu); \Phi)^{-1} \mu' \mathrm{LT}(\mathrm{o}(\mu''); \Phi) \kappa_1. \tag{16}$$

By Lemma 28(c) (β),

$$\kappa_1 \phi_1 \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu''); \Phi)^{-1} \theta' \theta_1 \mathrm{LT}(\mathrm{o}(\mu_{j_2}); \Phi) \kappa_0 \qquad \text{(see Fig. 79)}. \tag{17}$$

(Remember that $\mathrm{o}(\theta' \theta_1) = \mathrm{o}(\mu'')$ and $\mathrm{t}(\theta' \theta_1) = \mathrm{o}(\theta_2) = \mathrm{o}(\mu_{j_2})$.)

By 17°(β),

$$\kappa_0 \nu_1 \underset{i-1}{\simeq} \mathrm{LT}(\mathrm{o}(\mu_{j_2}); \Phi)^{-1} \nu_0 \eta_1^{-1}. \tag{18}$$

Comparing (16), (17) and (18), we obtain

$$\begin{aligned}
\omega' \tau_1 = \omega' \tau' \phi_1 \nu_1 &\underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu); \Phi)^{-1} \mu' \mathrm{LT}(\mathrm{o}(\mu''); \Phi) \kappa_1 \phi_1 \nu_1 \\
&\underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu); \Phi)^{-1} \mu' \theta' \theta_1 \mathrm{LT}(\mathrm{o}(\mu_{j_2}); \Phi) \kappa_0 \nu_1 \\
&\underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu); \Phi)^{-1} \mu' \theta' \theta_1 \nu_0 \eta_1^{-1} \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu); \Phi)^{-1} \mu_0 \eta_1^{-1}.
\end{aligned}$$

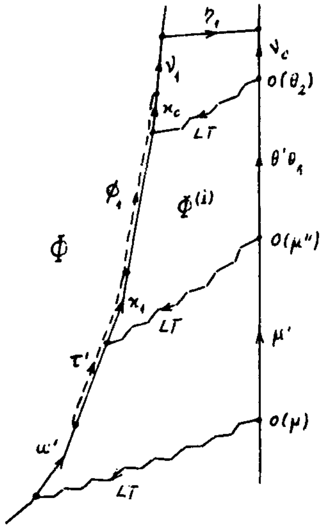


Fig. 79.

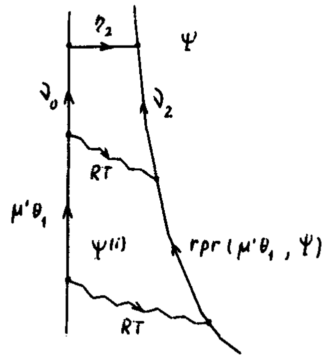


Fig. 80.

Thus, (A(β)) also holds.

We now verify (B). By 18°, $t(\eta)$ is a vertex on $pr(\mu_{i_2}; \Psi)$, hence on $pr(\mu; \Psi)$. Now let ω' be trivial. Then, by 9°, κ_1 is trivial, and then, by Lemma 28(g), θ' is trivial.

By 18°, ν_2 is a head of $pr(\mu_{i_2}; \Psi)$ such that $t(\eta) = t(\nu_2)$; hence the path $\tau_3 := rpr(\mu'\theta_1; \Psi)\nu_2$ is a head of $pr(\mu; \Psi)$ such that $t(\tau_3) = t(\eta)$ (see Fig. 80).

If θ_1 is trivial then, by 11° and 18°,

$$\tau_3 \in \mathcal{H}\left(\Psi; \sum_{j=1}^{i-1} 4\frac{1}{2}\cdot 13^{i-j}e_j + 4e_i\right) \subseteq \mathcal{H}\left(\Psi; \sum_{j=1}^{i} \frac{1}{2}\cdot 13^{i-j}e_j\right).$$

Since θ' is trivial, it follows from (9) and (12) that $\mu'' = \mu_1\mu_2\cdots\mu_h = \theta_1\theta_2\theta_3\theta''$. If θ_1 is non-trivial then, by Lemma 28(a), $\theta_1 = \mu_1$, and by Lemma 28(b), $\theta_1 = \mu_1 \not\subseteq S$. According to (11), μ_1 is not on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$. Then, by 12° and 18°,

$$\tau_3 = rpr(\mu'\theta_1; \Psi)\nu_2 \in \mathcal{H}\left(\Psi; \sum_{j=1}^{i-1} 5\frac{1}{2}\cdot 13^{i-j}e_j + 5e_i\right) \subseteq \mathcal{H}\left(\Psi; \sum_{j=1}^{i} \frac{1}{2}\cdot 13^{i-j}e_j\right).$$

We have thus verified (B(α)).

Next, by Lemma 15(d) and Lemma 26(a),

(19) $$rpr(\mu'\theta_1; \Psi) \underset{i}{\sim} RT(o(\mu); \Psi)^{-1}\mu'\theta_1 RT(o(\mu_{i_2}); \Psi).$$

On the other hand, by 18°,

(20) $$\nu_2 \underset{i-1}{\simeq} \mathrm{RT}(\mathrm{o}(\mu_{j_2});\Psi)^{-1}\nu_0\eta_2.$$

Comparing (19) and (20), we obtain

$$\tau_3 = \mathrm{rpr}(\mu'\theta_1;\Psi)\nu_2 \underset{i}{\sim} \mathrm{RT}(\mathrm{o}(\mu);\Psi)^{-1}\mu'\theta_1\mathrm{RT}(\mathrm{o}(\mu_{j_2});\Psi)\nu_2$$

$$\underset{i}{\sim} \mathrm{RT}(\mathrm{o}(\mu);\Psi)^{-1}\mu'\theta_1\nu_0\eta_2 = \mathrm{RT}(\mathrm{o}(\mu);\Psi)^{-1}\mu_0\eta_2,$$

hence (B(β)) is also verified. We have thus proved (C) in Case 1.

Case 2. $\phi_2 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ and ψ is trivial.

In this case, by (10) and (13), $\tau = \tau'\phi_1\phi_2\phi_3\tau'''$. Then, by 10°, 14° and 15°,

$$\tau = \tau'\phi_1\phi_2\phi_3\tau''' \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 7\cdot 13^{i-j}e_j + 6e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j\right);$$

therefore (C) holds.

Case 3. $\phi_2 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$, ψ is non-trivial, ω' is trivial and $\mathrm{rpr}(\theta_2;\Psi) \notin \mathcal{H}(\Psi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ (see Fig. 81).

By Definitions 17, 27 and 31, the fact that $\mathrm{rpr}(\theta_2;\Psi)$ is non-trivial implies that θ_2 is non-trivial. Then, by Lemma 28(c), $\theta_2 = \mu_{j_2} \in S$. Hence, by (11), $\theta_2 = \mu_{j_2}$ is on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$.

The path θ_2^{-1} is a non-trivial p.o.b.p. of $\Psi^{(i-1)}$ which is also a n.o.b.p. of $\Phi^{(i-1)}$. By Lemma 15(g), Lemma 26(a), (b) and Lemma 7,

$$\mathrm{pr}(\theta_2^{-1};\Psi) = \mathrm{pr}(\theta_2;\Psi)^{-1} = \mathrm{pr}(\mathrm{t}(\theta_2);\Psi)\mathrm{rpr}(\theta_2;\Psi)^{-1}.$$

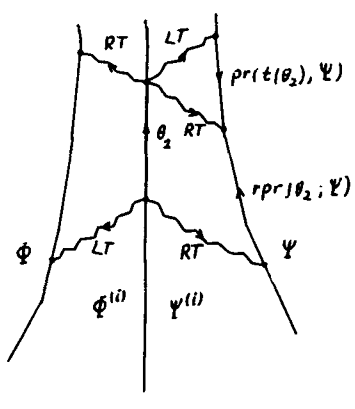


Fig. 81.

We apply the induction hypothesis with $i, \Phi, \Psi, \mu, \omega', \tau, \omega''$ replaced by $i-1$, $\Psi, \Phi, \theta_2^{-1}$, $\mathrm{pr}(\mathrm{t}(\theta_2); \Psi)$, $\mathrm{rpr}(\theta_2^{-1}; \Psi)$, $\mathrm{t}(\mathrm{rpr}(\theta_2^{-1}; \Psi))$. Since $\mathrm{rpr}(\theta_2^{-1}; \Psi) \notin \mathcal{H}(\Psi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$ it follows that there is a simple path $\chi \in \mathrm{Br}(i-1)$ connecting a vertex of $\mathrm{rpr}(\theta_2^{-1}; \Psi)$ to a vertex of $\mathrm{pr}(\theta_2^{-1}; \Phi)$ and having the following properties:

19°. Let ν_3 be the (minimal) tail of $\mathrm{rpr}(\theta_2^{-1}; \Psi)$ such that $\mathrm{o}(\nu_3) = \mathrm{o}(\chi)$. Then $\nu_3 \in \mathcal{H}(\Psi; \Sigma_{j=1}^{i-1} \frac{1}{2} \cdot 13^{i-j} e_j)$.

20°. There is a factorization $\chi = \chi_1 \chi_2$ such that:

(α) $\mathrm{t}(\chi_1) = \mathrm{o}(\chi_2)$ is a vertex on $\theta_2^{-1} = \mu_{i_2}^{-1}$, χ_1 is a path in $\mathrm{S}(\theta_2^{-1}; \Psi)$ and χ_2 is a path in $\mathrm{S}(\theta_2^{-1}; \Phi)$;

(β) if ν_4 is the tail of $\theta_2^{-1} = \mu_{i_2}^{-1}$ such that $\mathrm{o}(\nu_4) = \mathrm{t}(\chi_1)$, then

$$\nu_3 \underset{i-1}{\simeq} \chi_1 \nu_4 \mathrm{RT}(\mathrm{o}(\mu_{i_2}); \Psi);$$

(γ) if $\Phi < \Psi$, then χ_1 is trivial; if $\Psi < \Phi$ then χ_2 is trivial (see Fig. 82).

21°. $\mathrm{t}(\chi)$ is a vertex on $\mathrm{pr}(\theta_2^{-1}; \Phi)$. If ν_5 is the (minimal) tail of $\mathrm{pr}(\theta_2^{-1}; \Phi) = \mathrm{pr}(\theta_2; \Phi)^{-1} = \mathrm{pr}(\mu_{i_2}; \Phi)^{-1}$ such that $\mathrm{o}(\nu_5) = \mathrm{t}(\chi)$, then

$$\nu_5 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} \frac{1}{2} \cdot 13^{i-j} e_j\right) \quad \text{and} \quad \nu_5 \underset{i-1}{\simeq} \chi_2^{-1} \nu_4 \mathrm{LT}(\mathrm{o}(\mu_{i_2}); \Phi).$$

Take $\eta := \chi^{-1}$. We claim that (C) is satisfied.

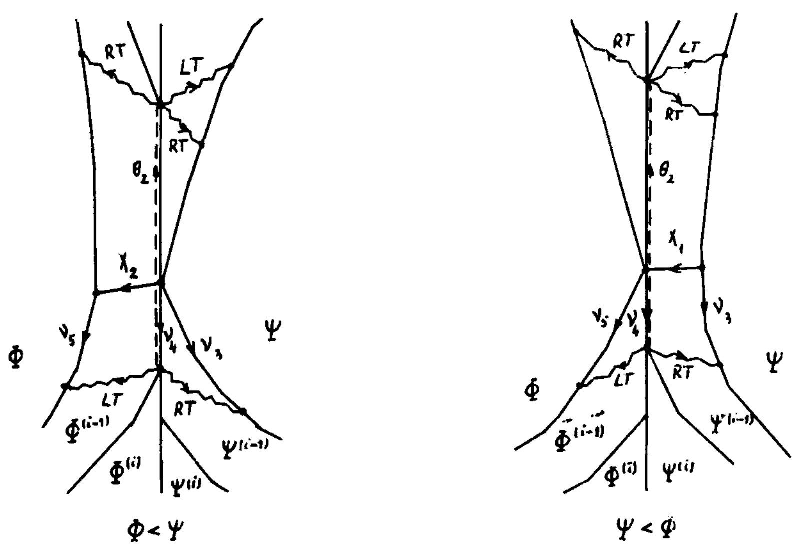


Fig. 82.

Indeed, since $\chi \in \mathrm{Br}(i-1)$, by Lemma 1(a), (c), $\eta \in \mathrm{Br}(i)$. Since ψ is non-trivial in Case 3, it follows from Lemma 28(e) that θ_3 is non-trivial and $\phi_3 = \mathrm{pr}(\theta_3;\Phi)$. Then, by 7°(β) and Lemma 28(d),

$$\omega'\tau'\phi_1\phi_2\phi_3 = \mathrm{lpr}(\mu';\Phi)\kappa_1\phi_1\phi_2\phi_3 = \mathrm{lpr}(\mu';\Phi)\mathrm{lpr}(\theta'\theta_1\theta_2;\Phi)\mathrm{pr}(\theta_3;\Phi)$$
$$= \mathrm{pr}(\mu'\theta'\theta_1\theta_2\theta_3;\Phi).$$

The vertex $\mathrm{o}(\eta) = \mathrm{t}(\chi)$ is on $\mathrm{pr}(\theta_2;\Phi)$, hence on $\mathrm{pr}(\mu'\theta'\theta_1\theta_2\theta_3;\Phi) = \omega'\tau'\phi_1\phi_2\phi_3$. But ω' is trivial in Case 3, and therefore $\mathrm{o}(\eta)$ is a vertex of $\tau'\phi_1\phi_2\phi_3$, hence of $\tau = \tau'\phi_1\phi_2\phi_3\psi\tau'''$ (cf. (2), (10) and (13)). On the other hand, $\mathrm{t}(\eta) = \mathrm{o}(\chi)$ is a vertex on $\mathrm{rpr}(\theta_2;\Psi)$, hence on $\mathrm{pr}(\mu;\Psi) = \mathrm{pr}(\mu'\theta'\theta_1\theta_2\theta_3\theta''\mu''';\Psi)$. Since χ is a simple path, $\eta = \chi^{-1}$ is also a simple path.

Since ω' is trivial, κ_1 is trivial by 9°, and then, by Lemma 28(g), θ' is trivial. Then, in view of (8) and (12) we have:

22°. $\mu'\theta_1$ is a head of μ such that $\mathrm{t}(\mu'\theta_1) = \mathrm{o}(\theta_2)$.

According to 21°, ν_5^{-1} is a head of $\mathrm{pr}(\theta_2;\Phi)$ such that $\mathrm{t}(\nu_5^{-1}) = \mathrm{t}(\chi) = \mathrm{o}(\eta)$. We have by Lemma 15 and Lemma 26(a)

$$\mathrm{t}(\mathrm{lpr}(\mu'\theta_1;\Phi)) = \mathrm{lpr}(\mathrm{t}(\mu'\theta_1);\Phi) = \mathrm{lpr}(\mathrm{o}(\theta_2);\Phi) = \mathrm{o}(\mathrm{pr}(\theta_2;\Phi)),$$

and so $\tau_1 := \mathrm{lpr}(\mu'\theta_1;\Phi)\nu_5^{-1}$ is a head of $\mathrm{pr}(\mu;\Phi)$ such that $\mathrm{t}(\tau_1) = \mathrm{o}(\eta)$ (see Fig. 83).

If θ_1 is trivial then, by Definitions 17, 27 and 32, $\mathrm{lpr}(\theta_1;\Phi)$ is also trivial. If θ_1 is non-trivial then, by Lemma 28(a), $\theta_1 = \mu_{j_1}$, and then, in view of (14), $\mathrm{lpr}(\theta_1;\Phi) \in \mathcal{H}(\Phi;\Sigma_{j=1}^{i} 13^{i-j}e_j)$. Using 11° and 21°, we obtain

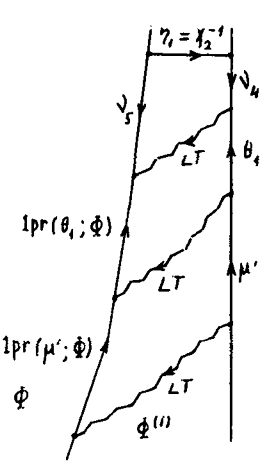


Fig. 83.

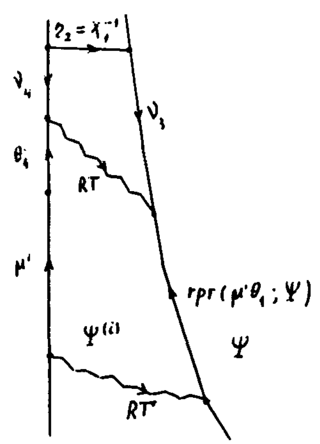


Fig. 84.

$$\tau_1 = \mathrm{lpr}(\mu'; \Phi)\mathrm{lpr}(\theta_1; \Phi)\nu_5^{-1} \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 5\frac{1}{2}\cdot 13^{i-j}e_j + 5e_i\right)$$

$$\subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

Since $\mathcal{M}$ satisfies (S_0), τ_1 cannot contain a boundary cycle of Φ and therefore τ_1 is the minimal head of τ such that $t(\tau_1) = o(\eta)$.

We now check condition (A).

Take $\eta_1 := \chi_2^{-1}$, $\eta_2 := \chi_1^{-1}$. Then, in view of 20°(α), $t(\eta_1) = o(\chi_2)$ is a vertex on μ_{j_2}, hence on μ. Since $\mu_{j_2} = \theta_2$ is a subpath of μ, we have also $S(\theta_2^{-1}; \Phi) \subseteq S(\mu; \Phi)$ and $S(\theta_2^{-1}; \Psi) \subseteq S(\mu; \Psi)$. Therefore 20°(α) implies (A(α)). (A(γ)) follows immediately from 20°(γ).

Define

$$\mu_0 := \mu'\theta_1\nu_4^{-1}. \tag{21}$$

In view of 20°(β) and 22°, μ_0 is the head of μ such that $t(\mu_0) = t(\chi_1) = o(\chi_2) = t(\eta_1)$. By Lemma 15 and Lemma 26(a),

$$\mathrm{lpr}(\mu'\theta_1; \Phi) \underset{i}{\sim} \mathrm{LT}(o(\mu); \Phi)^{-1}\mu'\theta_1 \mathrm{LT}(o(\mu_{j_2}); \Phi).$$

In view of 21° and the fact that in case 3 ω' is trivial,

$$\tau_1 = \mathrm{lpr}(\mu'\theta_1; \Phi)\nu_5^{-1} \underset{i}{\sim} \mathrm{LT}(o(\mu); \Phi)^{-1}\mu'\theta_1\nu_4^{-1}\chi_2 = \mathrm{LT}(o(\mu); \Phi)^{-1}\mu_0\eta_1^{-1},$$

so that (A(β)) is also verified.

We must now check condition (B).

The vertex $t(\eta) = o(\chi)$ is a vertex of $\mathrm{pr}(\theta_2; \Psi)$, hence of $\mathrm{pr}(\mu; \Psi)$. In view of 19° and 22°, $\tau_3 := \mathrm{rpr}(\mu'\theta_1; \Psi)\nu_3^{-1}$ is a head of $\mathrm{pr}(\mu; \Psi)$ such that $t(\tau_3) = o(\chi) = t(\eta)$ (see Fig. 84).

If θ_1 is trivial then by 11° and 19°,

$$\tau_3 = \mathrm{rpr}(\mu'; \Psi)\nu_3^{-1} \in \mathcal{H}\left(\Psi; \sum_{j=1}^{i-1} 4\frac{1}{2}\cdot 13^{i-j}e_j + 4e_i\right) \subseteq \mathcal{H}\left(\Psi; \sum_{j=1}^{i} \frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

Since θ' is trivial, it follows from (9) and (12) that θ_1 is a head of $\mu'' = \mu_1\mu_2\cdots\mu_h$. If θ_1 is non-trivial then, by Lemma 28(a), $\theta_1 = \mu_{j_1}$. By condition (α) of Lemma 28, each μ_j is non-trivial; therefore necessarily $j_1 = 1$ and then $\theta_1 = \mu_1$. By Lemma 28(b), $\theta_1 = \mu_1 \not\in S$, hence, by (11), μ_1 is not on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$. Then, by 12° and 19°,

$$\tau_3 = \operatorname{rpr}(\mu'\theta_1;\Psi)\nu_3^{-1} \in \mathcal{K}\left(\Psi;\sum_{j=1}^{i-1} 5\frac{1}{2}\cdot 13^{i-j}e_j + 5e_i\right) \subseteq \mathcal{K}\left(\Psi;\sum_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

This verifies (B(α)).

Now, by Lemma 15 and Lemma 26(a),

$$\operatorname{rpr}(\mu'\theta_1;\Psi) \underset{i}{\sim} \operatorname{RT}(o(\mu);\Psi)^{-1}\mu'\theta_1\operatorname{RT}(o(\mu_{j_2});\Psi).$$

Then, using 20°(β) and (21), we obtain

$$\tau_3 = \operatorname{rpr}(\mu'\theta_1;\Psi)\nu_3^{-1} \underset{i}{\sim} \operatorname{RT}(o(\mu);\Psi)^{-1}\mu'\theta_1\nu_4^{-1}\chi_1^{-1} = \operatorname{RT}(o(\mu);\Psi)^{-1}\mu_0\eta_2.$$

Thus, (B(β)) is also verified, and we have proved (C) in case 3.

Case 4. $\phi_2 \in \mathcal{K}(\Phi;\Sigma_{j=1}^{i-1}13^{i-j}e_j)$, ψ is non-trivial and either ω' is non-trivial or $\operatorname{rpr}(\theta_2;\Psi)\in\mathcal{K}(\Psi;\Sigma_{j=1}^{i-1}13^{i-j}e_j)$.

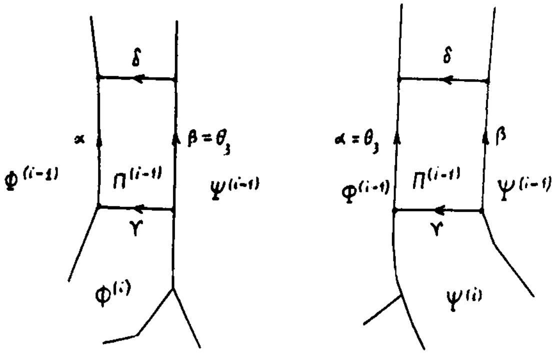
By Lemma 28(e), θ_3 and θ'' are non-trivial and $\phi_3 = \operatorname{pr}(\theta_3;\Phi)$. By Lemma 28(f), $\theta_1\theta_2$ is non-trivial. By Lemma 28(a), $\theta_3 = \mu_{j_3}$ and then, in view of (9) and (12), $1 < j_3 < h$. By Lemma 28(d), $\theta_3 = \mu_{j_3} \notin S$ and then, by (11), $\theta_3 = \mu_{j_3}$ is not on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$. Then, by Proposition 2(h) and (h′), we obtain

23°. If $\Phi < \Psi$, then $\theta_3 = \beta(\Gamma_{j_3}^{(i-1)})$ for some $\Gamma_{j_3}^{(i-1)} \in \mathcal{L}^1_{M^{(i-1)}}(\Phi^{(i-1)})$; if $\Psi < \Phi$, then $\theta_3 = \beta(\Pi_{j_3}^{(i-1)})^{-1}$ for some $\Pi_{j_3}^{(i-1)} \in \mathcal{L}^1_{M^{(i-1)}}(\Psi^{(i-1)})$.

In order to simplify the notation, we introduce the following abbreviations:

24°. If $\Phi < \Psi$, then $\Pi := \Gamma_{j_3}$, $\alpha := \alpha(\Gamma_{j_3}^{(i-1)})$, $\beta := \beta(\Gamma_{j_3}^{(i-1)}) = \theta_3$, $\gamma := \gamma(\Gamma_{j_3}^{(i-1)})$, $\delta := \delta(\Gamma_{j_3}^{(i-1)})$.

If $\Psi < \Phi$, then $\Pi := \Pi_{j_3}$, $\alpha := \beta(\Pi_{j_3}^{(i-1)})^{-1} = \theta_3$, $\beta := \alpha(\Gamma_{j_3}^{(i-1)})^{-1}$, $\gamma := \delta(\Pi_{j_3}^{(i-1)})^{-1}$, $\delta := \gamma(\Pi_{j_3}^{(i-1)})^{-1}$ (see Fig. 85).



$\Phi < \Psi$ $\Psi < \Phi$

Fig. 85.

Since $1 < j_3 < h$, Proposition 2(h), (h′) yields:

25°. α is on the common boundary of $\Phi^{(i-1)}$ and $\Pi^{(i-1)}$ while β is on the common boundary of $\Pi^{(i-1)}$ and $\Psi^{(i-1)}$. Furthermore, $\alpha = \mathrm{pr}(\theta_3; \Phi^{(i-1)})$ and $\beta = \mathrm{pr}(\theta_3; \Psi^{(i-1)})$.

By the assumption, the regions Φ and Ψ are of ranks r and s, respectively. Therefore, by the induction hypothesis,

$$\mathrm{pr}(\alpha;\Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j + e_r\right), \tag{22}$$

and

$$\mathrm{pr}(\beta;\Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j + e_s\right). \tag{23}$$

By Lemma 22(a), $\gamma \in \Pi^{(i-1)}(e_1)$ and $\delta \in \Pi^{(i-1)}(e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$, hence in $\mathcal{M}^{(i-1)}$. Then, by the induction hypothesis, Corollary 1 and 4°,

$$\mathrm{pr}(\gamma;\Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i} 13^{i-j} e_j\right),\ \mathrm{pr}(\delta;\Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i} 13^{i-j} e_j\right). \tag{24}$$

Suppose that $\mathrm{pr}(\alpha;\Pi) \in \mathcal{H}(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j)$. Then, by (23) and (24),

$$\mathrm{pr}(\alpha^{-1}\gamma^{-1}\beta\delta;\Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 4\cdot 13^{i-j} e_j + 2e_i + e_s\right).$$

But this contradicts (S_0), because by Lemma 6, Lemma 7(f) and Lemma 26(b) $\mathrm{pr}(\alpha^{-1}\gamma^{-1}\beta\delta;\Pi)$ contains a boundary cycle of Π. Therefore

$$\mathrm{pr}(\alpha;\Pi) \notin \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right). \tag{25}$$

Similarly,

$$\mathrm{pr}(\beta;\Pi) \notin \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right). \tag{26}$$

We now apply the induction hypothesis twice. The first application is with $i, \Phi, \Psi, \mu, \omega', \tau, \omega''$ replaced by $i-1$, Π, Φ, α^{-1}, $o(\mathrm{pr}(\alpha^{-1};\Pi))$, $\mathrm{pr}(\alpha^{-1};\Pi)$, $t(\mathrm{pr}(\alpha^{-1};\Pi))$ (see Fig. 86). By Lemma 15 and Lemma 26(b),

$$\mathrm{pr}(\alpha^{-1};\Pi) = \mathrm{pr}(\alpha;\Pi)^{-1}.$$

Then by (25) and Lemma 1(a),

$$\mathrm{pr}(\alpha^{-1};\Pi) \notin \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right).$$

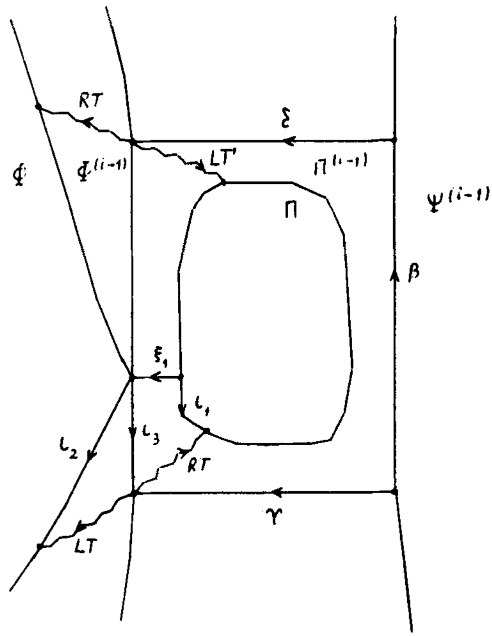


Fig. 86.

Therefore there is a simple path $\xi_1 \in \mathrm{Br}(i-1)$ connecting a vertex of $\mathrm{pr}(\alpha^{-1};\Pi)$ to a vertex of $\mathrm{pr}(\alpha^{-1};\Phi)$ and having the following properties:

26°. Let ι_1 be the (minimal) tail of $\mathrm{pr}(\alpha^{-1};\Pi)$ such that $\mathrm{o}(\iota_1)=\mathrm{o}(\xi_1)$. Then $\iota_1 \in \mathcal{H}(\Pi;\Sigma_{j=1}^{i-1}\frac{1}{2}\cdot 13^{i-j}e_j)$.

27°. Let ι_2 be the (minimal) tail of $\mathrm{pr}(\alpha^{-1};\Phi)$ such that $\mathrm{o}(\iota_2)=\mathrm{t}(\xi_1)$. Then $\iota_2 \in \mathcal{H}(\Phi;\Sigma_{j=1}^{i-1}\frac{1}{2}\cdot 13^{i-j}e_j)$.

28°. ξ_1 is a path in $\mathrm{S}(\alpha^{-1};\Pi)$ and $\mathrm{t}(\xi_1)$ is a vertex both on α^{-1} and $\mathrm{pr}(\alpha^{-1};\Phi)$. (Indeed, either $\Pi^{(i-1)} \in \mathcal{L}^1_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$ or $\Pi^{(i-1)} \in \mathcal{L}^1_{\mathcal{M}^{(i-1)}}(\Psi^{(i-1)})$. In both cases $\mathrm{rank}(\Pi)=i<r=\mathrm{rank}(\Phi)$ and we then apply (A′(γ)) of the induction hypothesis.)

29°. Let ι_3 be the tail of α^{-1} such that $\mathrm{o}(\iota_3)=\mathrm{t}(\xi_1)$. Then $\iota_2 \sim_{i-1} \iota_3\mathrm{LT}(\mathrm{o}(\alpha);\Phi)$.

We now apply the induction hypothesis again, with $i,\Phi,\Psi,\mu,\omega',\tau,\omega''$ replaced by $i-1,\Pi,\Psi,\beta$, $\mathrm{o}(\mathrm{pr}(\beta;\Pi))$, $\mathrm{pr}(\beta;\Pi)$, $\mathrm{t}(\mathrm{pr}(\beta;\Pi))$ (see Fig. 87).

In view of (26), there is a simple path $\xi_2 \in \mathrm{Br}(i-1)$ connecting a vertex of $\mathrm{pr}(\beta;\Pi)$ to a vertex of $\mathrm{pr}(\beta;\Psi)$ and having the following properties:

30°. Let ι_4 be the (minimal) head of $\mathrm{pr}(\beta;\Pi)$ such that $\mathrm{t}(\iota_4)=\mathrm{o}(\xi_2)$. Then $\iota_4 \in \mathcal{H}(\Pi;\Sigma_{j=1}^{i-1}\frac{1}{2}\cdot 13^{i-j}e_j)$.

31°. Let ι_5 be the (minimal) head of $\mathrm{pr}(\beta;\Psi)$ such that $\mathrm{t}(\iota_5)=\mathrm{t}(\xi_2)$. Then $\iota_5 \in \mathcal{H}(\Psi;\Sigma_{j=1}^{i-1}\frac{1}{2}\cdot 13^{i-j}e_j)$.

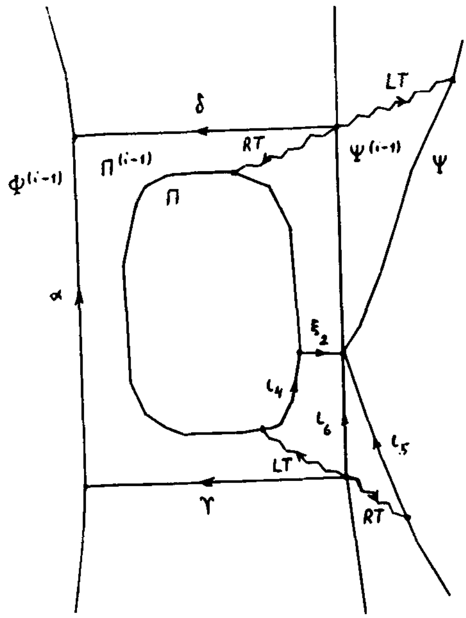


Fig. 87.

32°. ξ_2 is a path in $S(\beta;\Pi)$ and $t(\xi_2)$ is a vertex both on β and $pr(\beta;\Psi)$. (Here we are using the fact that $rank(\Pi) = i < s = rank(\Psi)$.)

33°. Let ι_6 be the head of β such that $t(\iota_6) = t(\xi_2)$. Then $\iota_5 \sim_{i-1} RT(o(\beta);\Psi)^{-1}\iota_6$.

We now construct the path η.

Let ι_0 be the boundary path of Π connecting the vertex $t(pr(\alpha^{-1};\Pi)) = rpr(o(\alpha);\Pi)$ to the vertex $o(pr(\beta;\Pi)) = lpr(o(\beta);\Pi)$ and such that

$$\iota_0 \underset{i-1}{\sim} RT(o(\alpha);\Pi)^{-1}\gamma^{-1}LT(o(\beta);\Pi) \qquad \text{(see Fig. 88)}.$$

By Lemma 7(d) and Lemma 26(b),

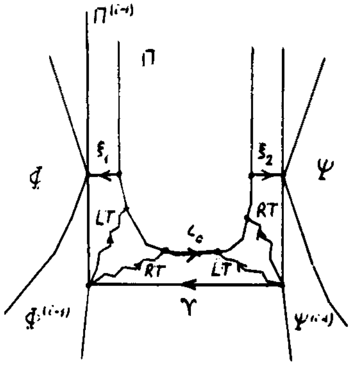
$$pr(\gamma^{-1};\Pi) = pr(t(\gamma);\Pi)rpr(\gamma^{-1};\Pi) = lpr(\gamma^{-1};\Pi)pr(o(\gamma);\Pi).$$

Therefore, either

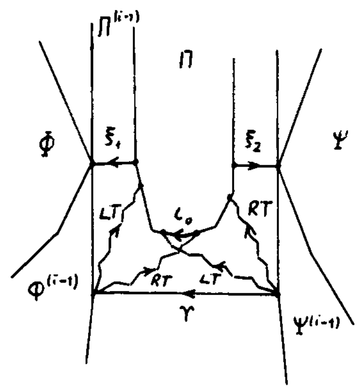
$$pr(\gamma^{-1};\Pi) = pr(t(\gamma);\Pi)\iota_0 pr(o(\gamma);\Pi)$$

or

$$pr(\gamma^{-1};\Pi) = lpr(\gamma^{-1};\Pi)\iota_0^{-1}rpr(\gamma^{-1};\Pi).$$



ι_0 is a p.o.b.p. of Π



ι_0 is a n.o.b.p. of Π

Fig. 88.

In each of these cases, it follows by (24) that

$$\iota_0 \in \mathscr{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right). \tag{27}$$

Now consider the boundary path ι of Π such that $o(\iota) = o(\iota_1) = o(\xi_1)$, $t(\iota) = t(\iota_4) = o(\xi_2)$ (see Figs. 86 and 87) and $\iota \sim_0 \iota_1 \iota_0 \iota_4$. In fact, ι is obtained by reducing, if necessary, the path $\iota_1 \iota_0 \iota_4$ (see Fig. 89). Then by 26°, 30° and (27)

$$\iota \in \mathscr{H}\left(\Pi; \sum_{j=1}^{i-1} 2\cdot 13^{i-j} e_j + e_i\right). \tag{28}$$

Let η be the path obtained from $\xi_1^{-1} \iota \xi_2$ by deleting all its closed subpaths (if there are any) (see Fig. 90).

We can now prove (C).

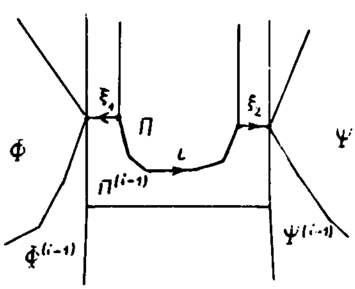


Fig. 89.

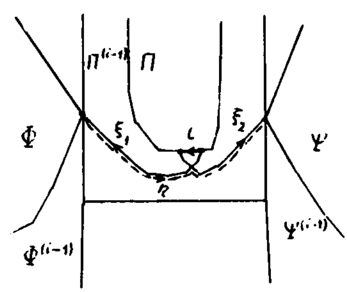


Fig. 90.

Indeed, we know that $\xi_1 \in \mathrm{Br}(i-1)$, $\xi_2 \in \mathrm{Br}(i-1)$ and $\iota \in \mathcal{H}(\Pi; \Sigma_{j=1}^{i-1} 2\cdot 13^{i-j}e_j + e_i)$. Therefore $\xi_1^{-1}\iota\xi_2$ belongs to $\mathrm{Br}(i)$ by Definition 9 and Lemma 1(a). (In fact, this point actually determined the definition of $\mathrm{Br}(i)$.) By Lemma 2, we have also $\eta \in \mathrm{Br}(i)$. By construction, η is a simple path.

In Case 4, ψ is non-trivial; hence by Lemma 28(e) and 25°,

$$\phi_3 = \mathrm{pr}(\theta_3;\Phi) = \mathrm{pr}(\mu_{j_3};\Phi) = \mathrm{pr}(\alpha;\Phi). \tag{29}$$

On the other hand, by 25°,

$$\mathrm{pr}(\theta_3;\Psi) = \mathrm{pr}(\mu_{j_3};\Psi) = \mathrm{pr}(\beta;\Psi). \tag{30}$$

In view of (10) and (13), ϕ_3 is a subpath of τ. By the construction of ξ_1, $\mathrm{o}(\eta) = \mathrm{t}(\xi_1)$ is a vertex of $\mathrm{pr}(\alpha;\Phi) = \phi_3$, hence of τ. By the construction of ξ_2, $\mathrm{t}(\eta) = \mathrm{t}(\xi_2)$ is a vertex of $\mathrm{pr}(\beta;\Psi)$, hence of $\mathrm{pr}(\mu;\Psi)$. Thus, η connects a vertex of τ to a vertex of $\mathrm{pr}(\mu;\Psi)$.

Using (10), (13), (29) and 27°, we see that the path τ_1 defined by

$$\tau_1 := \tau'\phi_1\phi_2\iota_2^{-1} \tag{31}$$

is a head of τ such that $\mathrm{t}(\tau_1) = \mathrm{t}(\xi_1) = \mathrm{o}(\eta)$. By 10°, 14°, 27° and the assumptions of Case 4,

$$\tau_1 = \tau'\phi_1\phi_2\iota_2^{-1} \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 4\frac{1}{2}\cdot 13^{i-j}e_j + 3e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

Since $\mathcal{M}$ satisfies (S_0), τ_1 cannot contain a boundary cycle of Φ and therefore τ_1 is the minimal head of τ such that $\mathrm{t}(\tau_i) = \mathrm{o}(\eta)$.

We now verify condition (A), under the assumption that $\Phi < \Psi$.

In this case we take $\eta_1 := \eta$, $\eta_2 := \mathrm{t}(\eta)$.

By 23° and 24°, $\beta = \theta_3 = \mu_{j_3}$. By (8) and (9), μ_{j_3} is a subpath of μ and then, by 32°, $\mathrm{t}(\eta_1) = \mathrm{o}(\eta_2) = \mathrm{t}(\eta) = \mathrm{t}(\xi_2)$ is a vertex of μ. By 23° and 24°, $\Pi^{(i-1)} \in \mathcal{L}^1_{k^{(i-1)}}(\Phi^{(i-1)})$, and therefore $\mathrm{clos}(\Pi^{(i-1)}) \subseteq \mathrm{supp}(\mathrm{S}(\beta;\Phi)) \subseteq \mathrm{supp}(\mathrm{S}(\mu;\Phi))$ (see Fig. 91). By 28°, 32° and the construction of ι, $\xi_1^{-1}\iota\xi_2$ is in $\mathrm{clos}(\Pi^{(i-1)})$, hence it is a path in $\mathrm{S}(\mu;\Phi)$, and then $\eta_1 = \eta$ is also a path in $\mathrm{S}(\mu;\Phi)$. On the other hand, $\eta_2 = \mathrm{t}(\eta) = \mathrm{t}(\xi_2)$ is a vertex of $\mathrm{pr}(\beta;\Psi)$, hence of $\mathrm{pr}(\mu;\Psi)$. Then, obviously, η_2 is a (trivial) path in $\mathrm{S}(\mu;\Psi)$. Furthermore, in view of (8), (9), (12) and 33°, the path μ_0 defined by

$$\mu_0 := \mu'\theta'\theta_1\theta_2\iota_6 \tag{32}$$

is the head of μ such that $\mathrm{t}(\mu_0) = \mathrm{t}(\xi_2) = \mathrm{t}(\eta) = \mathrm{t}(\eta_1)$.

By 7° (β), Lemma 15 and Lemma 26(a),

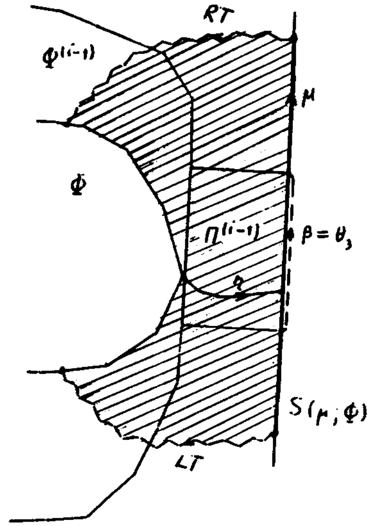


Fig. 91.

$$\omega'\tau' \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu'\mathrm{LT}(\mathrm{o}(\mu'');\Phi)\kappa_1. \tag{33}$$

By Lemma 28(d),

$$\kappa_1\phi_1\phi_2 \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu');\Phi)^{-1}\theta'\theta_1\theta_2\mathrm{LT}(\mathrm{o}(\mu_{t_3});\Phi). \tag{34}$$

By Definitions 16, 27 and 32,

$$\mathrm{LT}(\mathrm{o}(\mu_{t_3});\Phi) = \gamma\,\mathrm{LT}(\mathrm{o}(\alpha);\Phi). \tag{35}$$

By 29°, $\mathrm{o}(\eta) = \mathrm{t}(\xi_1) = \mathrm{o}(\iota_3)$ and by 33°, $\mathrm{t}(\eta) = \mathrm{t}(\xi_2) = \mathrm{t}(\iota_6)$. Since η is a path in $\mathrm{clos}(\Pi^{(i-1)})$,

$$\iota_3 \underset{i}{\sim} \eta\iota_6^{-1}\gamma. \tag{36}$$

By 29°,

$$\iota_2 \underset{i-1}{\sim} \iota_3\mathrm{LT}(\mathrm{o}(\alpha);\Phi) \qquad \text{(see Fig. 92)}. \tag{37}$$

Using (13), (32), (33), (34), (35), (36) and (37), we obtain

$$\begin{aligned}\omega'\tau_1 = \omega'\tau'\phi_1\phi_2\iota_2^{-1} &\underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu'\mathrm{LT}(\mathrm{o}(\mu'');\Phi)\kappa_1\phi_1\phi_1\iota_2^{-1}\\ &\underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu'\theta'\theta_1\theta_2\gamma\,\mathrm{LT}(\mathrm{o}(\alpha);\Phi)\iota_2^{-1}\\ &\underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu'\theta'\theta_1\theta_2\iota_6\eta_1^{-1}\iota_3\mathrm{LT}(\mathrm{o}(\alpha);\Phi)\iota_2^{-1} \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu_0\eta_1^{-1}.\end{aligned}$$

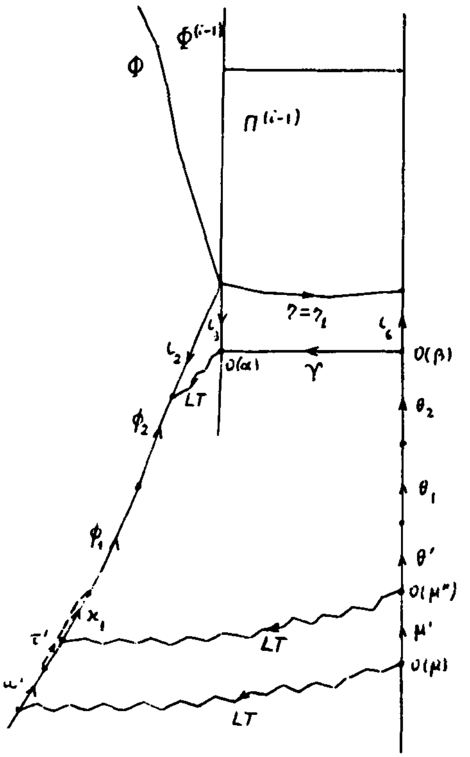


Fig. 92.

We have thus verified (A) under the assumption that $\Phi < \Psi$. We now verify (B) under the same assumption.

We have already shown that $t(\eta) = t(\xi_2)$ is a vertex of $\mathrm{pr}(\mu;\Psi)$. If ω' is trivial then, by 9°, κ_1 is trivial (remember that we assume that τ'' is non-trivial). Then, by Lemma 28(g), θ' is trivial. Define τ_3 by

$$(38) \qquad \tau_3 := \mathrm{rpr}(\mu'\theta_1\theta_2;\Psi)\iota_5.$$

We know that

$$t(\mathrm{rpr}(\mu'\theta_1\theta_2);\Psi) = \mathrm{rpr}(t(\theta_2);\Psi) = \mathrm{rpr}(o(\theta_3);\Psi) = o(\iota_5)$$

because, by 31°, ι_5 is a head of $\mathrm{pr}(\beta;\Psi) = \mathrm{pr}(\theta_3;\Psi)$ (see Fig. 93). Hence τ_3 is well-defined. By (8) and (9), $\mu'\theta_1\theta_2$ is a head of μ; therefore τ_3 is a head of $\mathrm{pr}(\mu;\Psi)$. By 31°, $t(\tau_3) = t(\iota_6) = t(\xi_2) = t(\eta)$. By the assumptions of Case 4, if ω' is trivial, then

$$(39) \qquad \mathrm{rpr}(\theta_2;\Psi) \in \mathcal{H}\left(\Psi; \sum_{j=1}^{i-1} 13^{i-j}e_j\right).$$

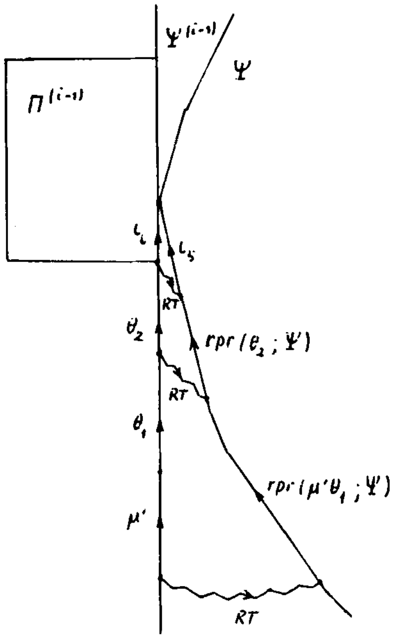


Fig. 93.

If θ_1 is trivial then, by 11°, (39) and 31°,

$$\tau_3 = \mathrm{rpr}(\mu'; \Psi)\mathrm{rpr}(\theta_2; \Psi)\iota_5 \in \mathcal{H}\left(\Psi; \sum_{j=1}^{i-1} 5\frac{1}{2}\cdot 13^{i-j}e_j + 4e_i\right) \subseteq \mathcal{H}\left(\Psi; \sum_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

If θ_1 is non-trivial then, since $\mu'' = \mu_1\mu_2\cdots\mu_h = \theta_1\theta_2\theta_3\theta''$, it follows from Lemma 28(a), (b) that $\theta_1 = \mu_1 \not\in S$. By (11), μ_1 is not on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$ and then, by 12°, (39) and 31°,

$$\tau = \mathrm{rpr}(\mu'\mu_1; \Psi)\mathrm{rpr}(\theta_2; \Psi)\iota_5 \in \mathcal{H}\left(\Psi; \sum_{j=1}^{i-1} 6\frac{1}{2}\cdot 13^{i-j}e_j + 5e_i\right)$$

$$\subseteq \mathcal{H}\left(\Psi; \sum_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

In view of (S_0), in both cases τ_3 does not contain a boundary cycle of Ψ, and therefore τ_3 is the minimal head of $\mathrm{pr}(\mu; \Psi)$ such that $\mathrm{t}(\tau_3) = \mathrm{t}(\eta)$. By (32) and 33°,

$$\tau_3 = \mathrm{rpr}(\mu'\theta_1\theta_2; \Psi)\iota_5 \underset{i}{\sim} \mathrm{RT}(\mathrm{o}(\mu); \Psi)^{-1}\mu'\theta_1\theta_2\mathrm{RT}(\mathrm{o}(\beta); \Psi)\iota_5$$

$$\underset{i}{\sim} \mathrm{RT}(\mathrm{o}(\mu); \Psi)^{-1}\mu'\theta_1\theta_2\iota_6 = \mathrm{RT}(\mathrm{o}(\mu); \Psi)^{-1}\mu_0 = \mathrm{RT}(\mathrm{o}(\mu); \Psi)^{-1}\mu_0\eta_2$$

because $\eta_2 = \mathrm{t}(\eta)$ is a trivial path.

Thus, (B) is also verified, under the assumption that $\Phi < \Psi$.

We now assume that $\Psi < \Phi$. Let us verify (A). Take $\eta_1 := o(\eta)$, $\eta_2 := \eta$.

By 23° and 24°, $\alpha = \theta_3 = \mu_{i_3}$; hence, by 28°, $t(\eta_1) = o(\eta) = t(\xi_1)$ is a vertex of μ_{i_3}, hence of μ. By 23° and 24°, $\Pi^{(i-1)} \in \mathcal{L}^1_{\mathcal{M}^{(i-1)}}(\Psi^{(i-1)})$, therefore $\xi_1^{-1}\iota\xi_2$ is a path in $S(\mu;\Psi)$, and then $\eta = \eta_2$ is a path in $S(\mu;\Psi)$ (see Fig. 94).

On the other hand, by the construction of ξ_1, $\eta_1 = o(\eta) = t(\xi_1)$ is a vertex of $pr(\alpha;\Phi)$, hence of $pr(\mu;\Phi)$ and then, of course, the (trivial) path η_1 is in $S(\mu;\Phi)$.

Furthermore, in view of (8), (9), (12) and 29°, the path

$$\mu_0 := \mu'\theta'\theta_1\theta_2\iota_3^{-1} \tag{40}$$

is a head of μ such that $t(\mu_0) = o(\iota_3) = t(\xi_1) = o(\eta) = o(\eta_2)$.

Using (31), (33), (34), (40) and 29° (all of which remain valid under the assumption that $\Psi < \Phi$), we obtain

$$\begin{aligned}\omega'\tau_1 = \omega'\tau'\phi_1\phi_2\iota_2^{-1} &\underset{i}{\sim} LT(o(\mu);\Phi)^{-1}\mu' LT(o(\mu'');\Phi)\kappa_1\phi_1\phi_2\iota_2^{-1}\\ &\underset{i}{\sim} LT(o(\mu);\Phi)^{-1}\mu'\theta'\theta_1\theta_2 LT(o(\alpha);\Phi)\iota_2^{-1}\\ &\underset{i}{\sim} LT(o(\mu);\Phi)^{-1}\mu'\theta'\theta_1\theta_2\iota_3^{-1} = LT(o(\mu);\Phi)^{-1}\mu_0\\ &= LT(o(\mu);\Phi)^{-1}\mu_0 o(\eta) = LT(o(\mu);\Phi)^{-1}\mu_0\eta_1^{-1} \qquad \text{(see Fig. 95).}\end{aligned}$$

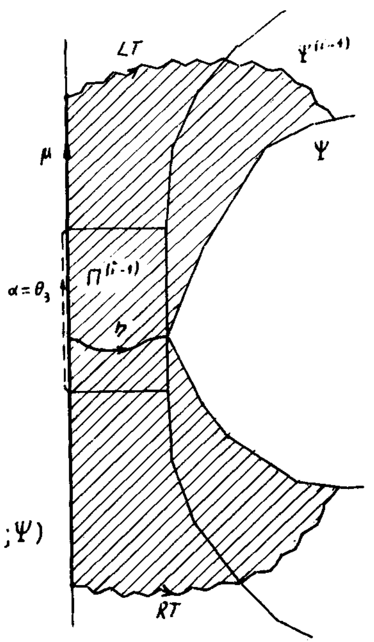


Fig. 94.

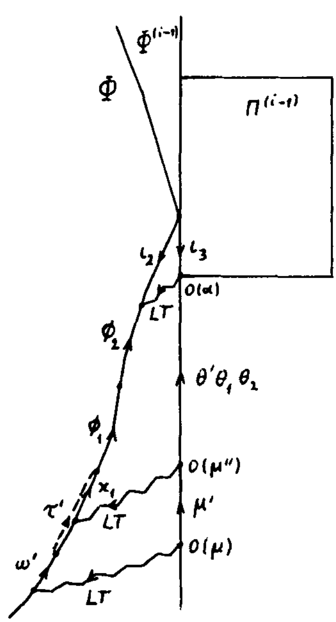


Fig. 95.

Thus parts (α) and (β) of (A) are verified. Part (γ) follows from the definition of η_1 and η_2.

Now consider (B). We already know that $t(\eta) = t(\xi_2)$ is a vertex of $\mathrm{pr}(\mu;\Psi)$. If ω' is trivial then, by 9°, κ_1 is trivial, and by Lemma 28(g), θ' is trivial. Then the path τ_3 defined by (38) is a head of $\mathrm{pr}(\mu;\Psi)$ such that $t(\tau_3) = t(\eta)$. Proceeding exactly as in the case $\Phi < \Psi$, we obtain $\tau_3 \in \mathcal{H}(\Psi; \Sigma_{j=1}^{i} \frac{1}{2}\cdot 13^{i+1-j} e_j)$, and then, in view of (S_0), τ_3 is the minimal head of $\mathrm{pr}(\mu;\Psi)$ such that $t(\tau_3) = t(\eta)$. Proceeding as in the derivation of (35), we have

$$\mathrm{RT}(o(\alpha);\Psi) = \gamma^{-1}\mathrm{RT}(o(\beta);\Psi).$$

Then, using (36), (38), (4) and 33° (see Fig. 96) (notice that (36) is valid under the assumption that $\Psi < \Phi$), we obtain

$$\tau_3 = \mathrm{rpr}(\mu'\theta_1\theta_2;\Psi)\iota_5 \underset{i}{\sim} \mathrm{RT}(o(\mu);\Psi)^{-1}\mu'\theta_1\theta_2\mathrm{RT}(o(\alpha);\Psi)\iota_5$$

$$\underset{i}{\sim} \mathrm{RT}(o(\mu);\Psi)^{-1}\mu'\theta_1\theta_2\gamma^{-1}\mathrm{RT}(o(\beta);\Psi)\iota_5 \underset{i}{\sim} \mathrm{RT}(o(\mu);\Psi)^{-1}\mu'\theta_1\theta_2\iota_3\eta$$

$$= \mathrm{RT}(o(\mu);\Psi)^{-1}\mu_0\eta = \mathrm{RT}(o(\mu);\Psi)^{-1}\mu_0\eta_2.$$

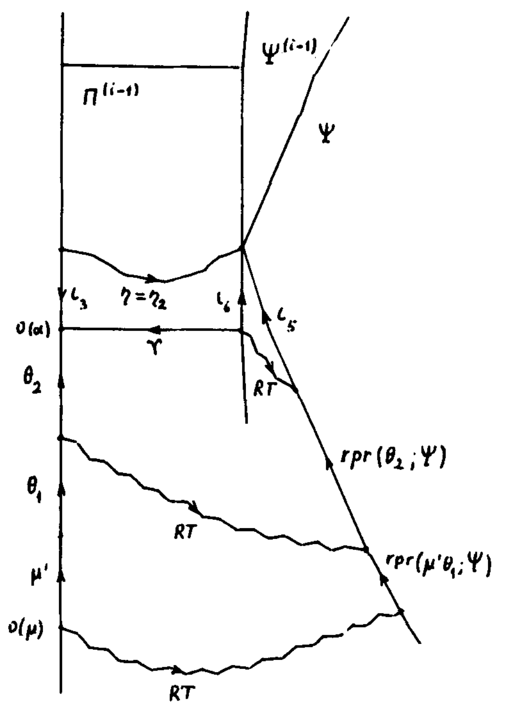


Fig. 96.

Thus, (B) is also verified under the assumption that $\Psi < \Phi$. This completes the proof of (C) in Case 4. Since Cases 1, 2, 3, 4 exhaust all possibilities, (C) is proved.

The following statement is proved in similar fashion.

(C)′ Either $\tau \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j} e_j)$, or there is a simple path $\eta' \in \mathrm{Br}(i)$ connecting a vertex of τ to a vertex of $\mathrm{pr}(\mu;\Psi)$, having properties (A′), (B′) and such that, if τ_2 is the (minimal) tail of τ satisfying $\mathrm{o}(\tau_2) = \mathrm{o}(\eta')$, then $\tau_2 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j)$.

Using (C) and (C′), we now easily complete the proof of the theorem. Indeed, if $\tau \notin \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j} e_j)$ then, by (C) and (C′), $\tau = \tau_1 \tau_1' = \tau_2' \tau_2$ for some τ_1' and τ_2'. If τ_1' is a tail of τ_2, then τ_1' belongs to $\mathcal{H}(\Phi; \Sigma_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j)$; then $\tau = \tau_1 \tau_1' \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j} e_j)$, contradicting our assumption. Therefore there is a path θ such that $\tau = \tau_1 \theta \tau_2$ (see Fig. 97).

By (A) and (A′), $\mathrm{t}(\eta_1)$ and $\mathrm{t}(\eta_1')$ are vertices of μ. Let ξ_0 be the subpath of μ or μ^{-1} connecting $\mathrm{t}(\eta_1)$ to $\mathrm{t}(\eta_1')$ (see Fig. 97). By Lemma 15 and Lemma 26(a),

$$\omega' \tau_1 \theta \tau_2 \omega'' = \omega' \tau \omega'' = \mathrm{pr}(\mu;\Phi) \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1} \mu \, \mathrm{RT}(\mathrm{t}(\mu);\Phi). \tag{41}$$

Using (A) and (A′), we have

$$\omega' \tau_1 \underset{i}{\sim} \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1} \mu_0 \eta_1^{-1}, \qquad \tau_2 \omega'' \underset{i}{\sim} \eta_1' \mu_0' \mathrm{RT}(\mathrm{t}(\mu);\Phi). \tag{42}$$

Comparing (41) and (42), we obtain

$$\theta \underset{i}{\sim} \eta_1 \xi_0 \eta_1'^{-1}. \tag{43}$$

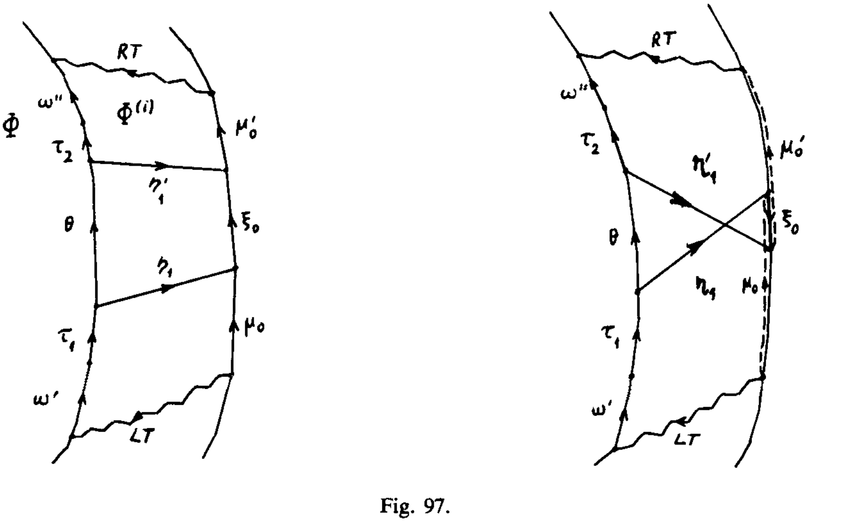


Fig. 97.

Let ξ be the boundary path of Ψ connecting $t(\eta)=t(\eta_2)$ to $t(\eta')=t(\eta_2')$ and such that $\xi_0 \sim_i \eta_2\xi\eta_2'^{-1}$ (see Fig. 98). By (A) and (A′), η_2 and η_2' are paths in $S(\mu;\Psi)$; hence the path ξ indeed exists. We obtain:

$$\theta \underset{i}{\sim} \eta_1\xi_0\eta_1'^{-1} \underset{i}{\sim} \eta_1\eta_2\xi\eta_2'^{-1}\eta_1'^{-1} = \eta\xi\eta'^{-1}.$$

By (C) and (C′), the paths η, η' are simple paths and belong to $\mathrm{Br}(i)$. Then, by Definition 9,

$$\theta \in \mathcal{P}(\Phi; s) = \mathcal{I}(\Phi; e_s).$$

We have $\tau_1, \tau_2 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} \frac{1}{2}\cdot 13^{i+1-j}e_j) \subseteq \mathcal{I}(\Phi; \Sigma_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j)$; hence, by Definition 9,

$$\tau = \tau_1\theta\tau_2 \in \mathcal{I}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j + e_s\right).$$

Since τ is an arbitrary subpath of $\mathrm{pr}(\mu;\Phi)$, it follows from Definition 9 that $\mathrm{pr}(\mu;\Phi) \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j}e_j + e_s)$.

This completes the proof of Theorem 4.

§7. Some modifications of Theorem 4

7.1. THEOREM 5. *Let $\mathcal{M} = (M, \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}, <)$ be an ordered n-ranked map satisfying condition* (S_0). *Let k be an integer, $0 \leqq k < n$. Assume that if $k > 0$ then $\mathcal{M}$ satisfies condition* (SC_k). *Let N be a regular k-submap such that* $\mathrm{int}(N)$ *is connected* (see Definitions 6 and 33). *Let m be the maximal integer such that $\mathcal{T}_m \cap \mathrm{Reg}(N) \neq \varnothing$. Then, of course,*

$$\mathcal{N} = (N, \{\mathcal{T}_1 \cap \mathrm{Reg}(N), \mathcal{T}_2 \cap \mathrm{Reg}(N), \cdots, \mathcal{T}_m \cap \mathrm{Reg}(N)\}, <)$$

is an ordered m-ranked map satisfying (S_0).

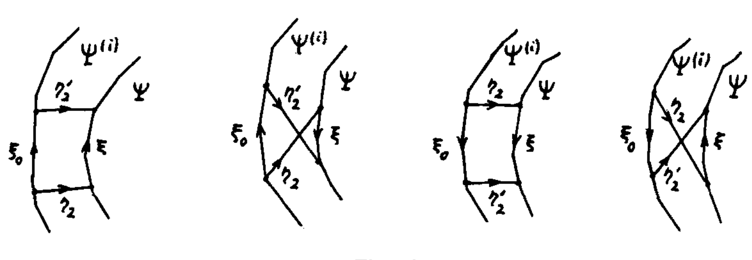


Fig. 98.

We shall use the subscript $\mathcal{M}$ to indicate constructions (projections, transversals, etc.) for which $\mathcal{M}$ is the underlying map. For all other constructions the underlying map is $\mathcal{N}$. An exception is made for notation of the type $\mathcal{H}(\Gamma; c)$ and $\lambda \sim_j \mu$ where the underlying map is always $\mathcal{M}$.

We assume that $\mathcal{N}$ satisfies (SC_i) for some $i, k \leqq i < n$.

Let Φ be a region of N, of rank $r > i$, and Ψ a region of M, of rank $s > i$; assume that $\Phi \neq \Psi$. Let μ be a positively oriented boundary path of $\Phi^{(i)} \in \mathrm{Reg}(N^{(i)})$ which is also a negatively oriented boundary path of $\Psi_{\mathcal{M}}^{(k)} \in \mathrm{Reg}(M^{(k)})$. Then

$$\mathrm{pr}(\mu; \Phi) \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j + e_s\right). \tag{1}$$

Moreover, let τ be a subpath of $\mathrm{pr}(\mu; \Phi)$, i.e. for some ω', ω'',

$$\mathrm{pr}(\mu; \Phi) = \omega' \tau \omega''. \tag{2}$$

Then either

$$\tau \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j\right) \tag{3}$$

or there is a factorization

$$\tau = \tau_1 \theta \tau_2 \tag{4}$$

such that

$$\tau_1, \tau_2 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j\right) \tag{5}$$

and

$$\theta \in \mathcal{I}_{\mathcal{M}}(\Phi; e_s) = \mathcal{P}_{\mathcal{M}}(\Phi; s). \tag{6}$$

More precisely, there are two simple paths (see Fig. 99) $\eta, \eta' \in \mathrm{Br}_{\mathcal{M}}(i)$ and a boundary path ξ of Ψ such that

$$\theta \underset{i}{\sim} \eta \xi \eta'^{-1} \tag{7}$$

where η and η' have the following additional properties:

(A) There exists a factorization $\eta = \eta_1 \eta_2$ such that

(α) $t(\eta_1) = o(\eta_2)$ is a vertex on μ, η_1 is a path in $S(\mu; \Psi)$ and η_2 a path in $S(\mu; \Psi)$;

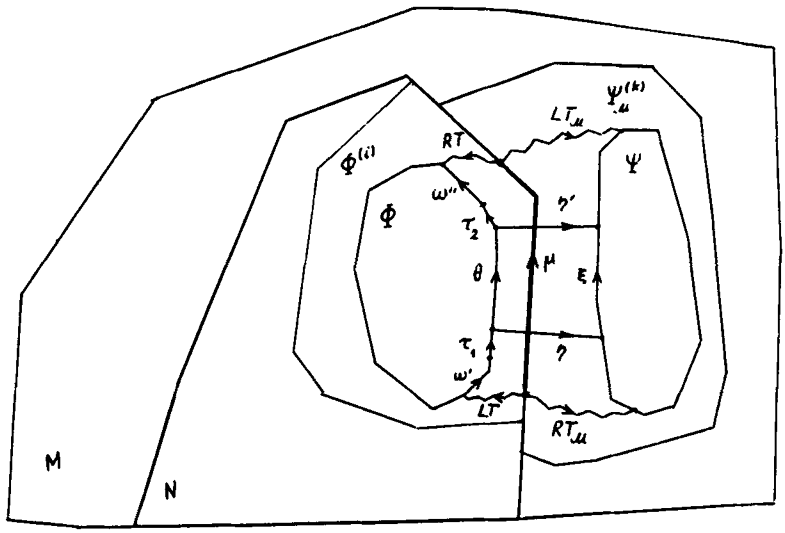


Fig. 99.

(β) if μ_0 is the head of μ such that $t(\mu_0) = t(\eta_1)$, then

$$\omega'\tau_1 \underset{i}{\sim} LT(o(\mu);\Phi)^{-1}\mu_0\eta_1^{-1};$$

(γ) if $\Phi < \Psi$, then η_2 is trivial; if $\Psi < \Phi$, then at least one of the paths η_1, η_2 is trivial (see Fig. 100).

(A′) There exists a factorization $\eta' = \eta_1'\eta_2'$ such that

(α) $t(\eta_1') = o(\eta_2')$ is a vertex on μ, η_1' is a path in $S(\mu;\Phi)$ and η_2' a path in $S(\mu;\Psi)$;

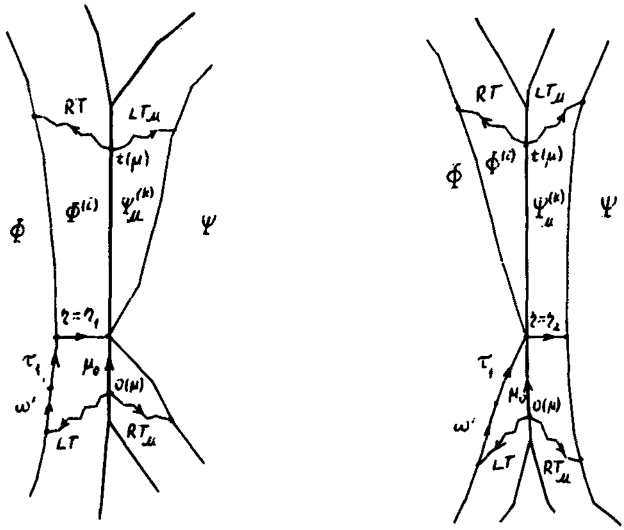
(β) if μ_0' is the tail of μ such that $o(\mu_0') = t(\eta_1')$, then

$$\tau_2\omega'' \underset{i}{\sim} \eta_1'\mu_0' RT(t(\mu);\Phi);$$

(γ) if $\Phi < \Psi$, then η_2' is trivial; if $\Psi < \Phi$, then at least one of the paths η_1', η_2' is trivial.

Proof. We proceed by induction on $i - k$.

If $i - k = 0$ then, by Lemma 27, $N^{(k)}$ is a submap of $M^{(k)} = M^{(i)}$ and all the constructions (projections, transversals etc.) in $\mathcal{N}^{(0)} = \mathcal{N}, \mathcal{N}^{(1)}, \cdots, \mathcal{N}^{(k)}$ based on $\mathcal{N}$ as underlying map are the same as those based on $\mathcal{M}$. In this case Theorem 5 follows from Theorem 4.



η_2 is trivial η_1 is trivial

Fig. 100.

Now let $i-k>0$.

Since (SC_i) implies (SC_l) for any $l<i$, the induction hypothesis implies:

1°. All the assertions of Theorem 5 hold whenever i is replaced by any $l, k \leqq l < i$.

Using the induction hypothesis, we obtain:

2°. Let Γ be a region of N, of rank i, such that $\Gamma^{(i-1)} \in \mathscr{L}_{\bar{\mu}^{(i-1)}}(\Phi^{(i-1)})$. Let σ be a subpath of μ which is a boundary path of $\Gamma^{(i-1)}$ (see Fig. 101). Then

$$\mathrm{pr}(\sigma;\Gamma) \in \mathscr{H}\left(\Gamma; \sum_{j=1}^{i-1} 13^{i-j} e_j + e_s\right).$$

Furthermore, we have

3°. Under the assumptions of 2°, if $d_{N^{(i-1)}}(\Gamma^{(i-1)}, \Phi^{(i-1)}) > 1$, then $\sigma \neq \beta(\Gamma^{(i-1)})$.

Let $\alpha := \alpha(\Gamma^{(i-1)})$, $\beta := \beta(\Gamma^{(i-1)})$, $\gamma := \gamma(\Gamma^{(i-1)})$, $\delta := \delta(\Gamma^{(i-1)})$. By Lemma 6 and Definition 26, $\alpha^{-1}\gamma^{-1}\beta\delta$ is a boundary cycle of $\Gamma^{(i-1)}$. By 2° of Theorem 4, $\tilde{\mathcal{N}}^{(i-1)}$ satisfies D(8) and D(6; 1). Then, by Lemma 22, $\delta\alpha^{-1}\gamma \in \Gamma^{(i-1)}(4e_1)$ in $\tilde{\mathcal{N}}^{(i-1)}$, hence in $\mathcal{N}^{(i-1)}$. By Lemma 7(d), (f) and Lemma 22(b), we can find a boundary cycle $\nu_1\nu_2$ of Γ such that ν_1 is a subpath of $\mathrm{pr}(\beta;\Gamma)$ and ν_2 is a subpath of $\mathrm{pr}(\delta\alpha^{-1}\gamma;\Gamma)$ (see Fig. 102). By Corollary 1 of Theorem 4,

$$\nu_2 \in \mathscr{H}\left(\Gamma; \sum_{j=1}^{i} 4\cdot 13^{i-j} e_j\right).$$

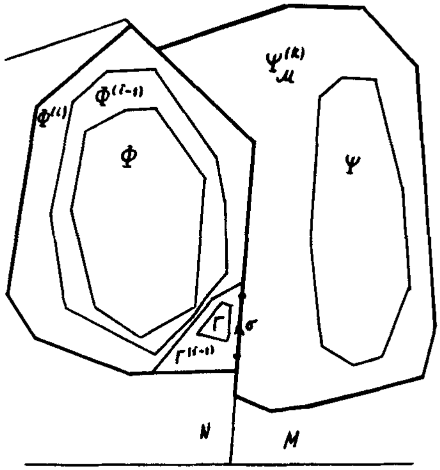


Fig. 101.

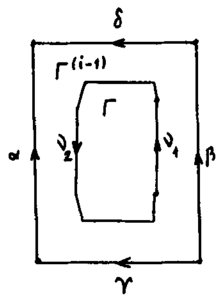


Fig. 102.

Now, if $\beta=\sigma$ then, by 2°, $\nu_1 \in \mathcal{H}(\Gamma;\Sigma_{j=1}^{i-1} 13^{i-j}e_j+e_s)$ and

$$\nu_1\nu_2 \in \mathcal{H}\left(\Gamma;\sum_{j=1}^{i-1} 5\cdot 13^{i-j}e_j+4e_i+e_s\right),$$

contradicting (S_0). Therefore $\sigma\neq\beta=\beta(\Gamma^{(i-1)})$, as required.

We now prove the following statement:

(C) Either $\tau\in\mathcal{H}(\Phi;\Sigma_{j=1}^{i} 13^{i+1-j}e_j)$, or there is a simple path $\eta\in \mathrm{Br}_{\mathcal{M}}(i)$, connecting a vertex of τ to a vertex of $\mathrm{pr}_{\mathcal{M}}(\mu;\Psi)$, having property (A), and such that, if τ_1 is the (minimal) head of τ with $t(\tau_1)=o(\eta)$, then $\tau_1\in\mathcal{H}(\Phi;\Sigma_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j)$.

Applying Proposition 1 with M,Φ,Φ' replaced by $N^{(i-1)}$, $\Phi^{(i-1)}$, $\Phi^{(i)}$, we obtain a factorization

(8) $$\mu = \mu'\mu''\mu'''$$

and, if μ'' is non-trivial, a further factorization

(9) $$\mu'' = \mu_1\mu_2\cdots\mu_h$$

such that

4°. μ' is a head of $\mathrm{RT}(\mathrm{o}(\mu);\Phi^{(i-1)})$.

5°. μ'''^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi^{(i-1)})$.

6°. If μ'' is non-trivial, then

(α) μ'' is on the boundary of $(\Phi^{(i-1)})^1$ (cf. Definition 23);

(β) the factorization (9) is the l.h.s. factorization of μ'' in $N^{(i-1)}$;

(γ) for any $j, 1 \leqq j \leqq h$, if μ_j is not on the boundary of $\Phi^{(i-1)}$, then $\mu_j = \beta(\Pi_j^{(i-1)})$ for some $\Pi_j^{(i-1)} \in \mathscr{L}^1_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})$.

As in the proof of Theorem 4 we conclude that there is a factorization

(10) $$\tau = \tau'\tau''\tau'''$$

with the following properties:

7°. If τ' (τ'', τ''') is non-trivial, it is a subpath of $\mathrm{pr}(\mu';\Phi)$ (of $\mathrm{pr}(\mu'';\Phi)$, of $\mathrm{pr}(\mu''';\Phi)$). Moreover, there are paths κ_1, κ_2 such that

(α) $\mathrm{pr}(\mu'';\Phi) = \kappa_1\tau''\kappa_2$;

(β) $\mathrm{lpr}(\mu';\Phi)\kappa_1 = \omega'\tau'$;

(γ) $\kappa_2\mathrm{rpr}(\mu''';\Phi) = \tau'''\omega''$ (see Fig. 76).

8°. If μ'' is trivial then τ'' is trivial.

As in the proof of Theorem 4, we obtain

9°. $\tau' \in \mathscr{H}(\Phi; \Sigma_{j=1}^i 2\cdot 13^{i-j}e_j)$ and $\tau''' \in \mathscr{H}(\Phi; \Sigma_{j=1}^i 2\cdot 13^{i-j}e_j)$.

Using (10), we have:

10°. If τ'' is trivial then $\tau = \tau'\tau''' \in \mathscr{H}(\Phi; \Sigma_{j=1}^i 4\cdot 13^{i-j}e_j) \subseteq \mathscr{H}(\Phi; \Sigma_{j=1}^i 13^{i+1-j}e_j)$.

In what follows, we assume that τ'' is non-trivial; then, by 8°, μ'' is also non-trivial.

Let S be the subset of $\{\mu_1, \mu_2, \cdots, \mu_h\}$ defined as follows:

(11) $$S := \{\mu_j \mid \mu_j \text{ is on the boundary of } \Phi^{(i-1)}\}.$$

Using Lemma 17(a) and 6°(β), we obtain that the paths μ_{j-1} and μ_j cannot both belong to S. We apply Lemma 28 with M, μ, ν replaced by N, μ'' and τ''. There result factorizations

(12) $$\mu'' = \theta'\theta_1\theta_2\theta_3\theta''$$

and

(13) $$\tau'' = \phi_1\phi_2\phi_3\psi$$

with the properties described in Lemma 28.

As in the proof of Theorem 4, one shows that

11°. $\phi_1, \phi_3 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} 13^{i-j}e_j)$.

The only change is that the reference to Proposition 2(h), (h′) is replaced by a reference to 6°(γ).

We have the following possibilities:

(1) $\phi_2 \notin \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$;

(2) $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ and ψ is trivial;

(3) $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ and ψ is non-trivial.

We consider each of these cases separately.

Case 1. $\phi_2 \notin \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$.

In this case ϕ_2 is non-trivial. Hence, by Lemma 28(c), θ_2 is non-trivial, and then

(14) $$\theta_2 = \mu_{i_2} \in S.$$

By (11), μ_{i_2} is on the boundary of $\Phi^{(i-1)}$. By Lemma 28(c), there are paths κ_0, κ_0' for which $\mathrm{pr}(\mu_{i_2}; \Phi) = \kappa_0\phi_2\kappa_0'$.

We apply the induction hypothesis with $i, \mu, \omega', \tau, \omega''$ replaced by $i-1, \mu_{i_2}, \kappa_0, \phi_2, \kappa_0'$. Since $\phi_2 \notin \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$, it follows that there is a simple path $\eta \in \mathrm{Br}_{\mathscr{M}}(i-1)$, connecting a vertex of ϕ_2 to a vertex of $\mathrm{pr}_{\mathscr{M}}(\mu_{i_2}; \Psi)$, and having the following properties:

12°. Let χ_1 be the (minimal) head of ϕ_2 such that $\mathrm{t}(\chi_1) = \mathrm{o}(\eta)$. Then $\chi_1 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} \frac{1}{2}\cdot 13^{i-j}e_j)$.

13°. There is a factorization $\eta = \eta_1\eta_2$ such that

(α) $\mathrm{t}(\eta_1) = \mathrm{o}(\eta_2)$ is a vertex of $\mu_{i_2} = \theta_2$, η_1 is a path in $\mathrm{S}(\mu_{i_2}; \Phi)$ and η_2 a path in $\mathrm{S}(\mu_{i_2}; \Psi)$;

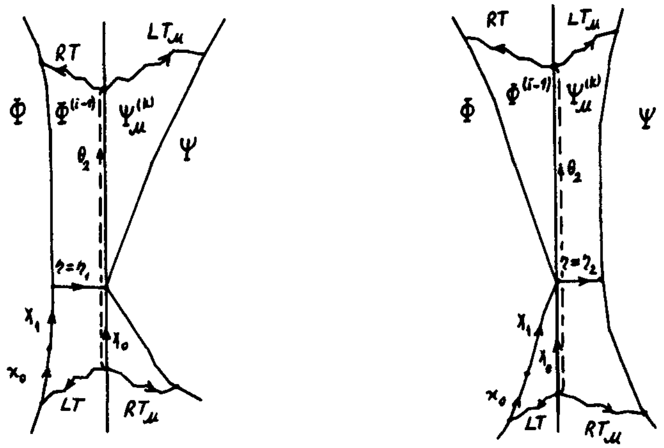
(β) if χ_0 is the head of μ_{i_2} such that $\mathrm{t}(\chi_0) = \mathrm{t}(\eta_1)$, then $\kappa_0\chi_1 \sim_{i-1} \mathrm{LT}(\mathrm{o}(\mu_{i_2}); \Phi)^{-1}\chi_0\eta_1^{-1}$;

(γ) if $\Phi < \Psi$ then η_2 is trivial; if $\Psi < \Phi$, then at least one of the paths η_1, η_2 is trivial (see Fig. 103).

We can now prove (C).

By (10) and (13), ϕ_2 is a subpath of τ. By (9), μ_{i_2} is a subpath of μ, hence $\mathrm{pr}_{\mathscr{M}}(\mu_{i_2}; \Psi)$ is a subpath of $\mathrm{pr}_{\mathscr{M}}(\mu; \Psi)$. By Lemma 1(c), $\mathrm{Br}_{\mathscr{M}}(i-1) \subseteq \mathrm{Br}_{\mathscr{M}}(i)$. Therefore, η is a simple path belonging to $\mathrm{Br}_{\mathscr{M}}(i)$ and connecting a vertex of τ to a vertex of $\mathrm{pr}_{\mathscr{M}}(\mu; \Psi)$. Define

(15) $$\tau_1 := \tau'\phi_1\chi_1.$$



η_2 is trivial $\qquad$ η_1 is trivial

Fig. 103.

Then, by (10), (13) and 12°, τ_1 is a head of τ such that $t(\tau_1) = o(\eta)$. By 9°, 11° and 12°,

$$\tau_1 = \tau'\phi_1\chi_1 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 3\frac{1}{2}\cdot 13^{i-j}e_j + 3e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

Since $\mathcal{M}$ satisfies (S_0), τ_1 cannot contain a boundary cycle of Φ and therefore τ_1 is the minimal head of τ such that $t(\tau_1) = o(\eta)$.

(A(α)) and (A(γ)) follow from 13°(α) and 13°(γ), respectively. We verify (A(β)). Denote

$$\mu_0 = \mu'\theta'\theta_1\chi_0. \tag{16}$$

By (8), (12) and 13°, μ_0 is the head of μ such that $t(\mu_0) = t(\eta_1)$. Using (15), (16), 7°(β), 13°(β), Lemma 28(c), Lemma 15(c) and Lemma 26(a), we conclude as in the proof of Theorem 4 that

$$\omega'\tau_1 \underset{i}{\sim} \mathrm{LT}(o(\mu);\Phi)^{-1}\mu_0\eta_1^{-1}$$

(see Fig. 104). Thus (A(β)) also holds.

This proves (C) in Case 1.

Case 2. $\phi_2 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ and ψ is trivial.

In this case we have by (10), (13), 9° and 11°,

$$\tau = \tau'\tau''\tau''' = \tau'\phi_1\phi_2\phi_3\tau''' \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 7\cdot 13^{i-j}e_j + 6e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j\right).$$

Thus (C) is true.

Case 3. $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$ and ψ is non-trivial.

Since ψ is non-trivial, it follows from Lemma 28(a), (d), (e) that $\theta_3 = \mu_{j_3} \notin S$ and $\phi_3 = \mathrm{pr}(\mu_{j_3}; \Phi)$. By (11) and 6°, there is a region $\Pi^{(i-1)} \in \mathscr{L}^1_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$ such that $\mu_{j_3} = \beta(\Pi^{(i-1)})$. Then, by Definitions 19, 26, 27 and 32:

14°. $\phi_3 = \mathrm{pr}(\beta(\Pi^{(i-1)}); \Phi) = \mathrm{pr}(\alpha(\Pi^{(i-1)}); \Phi)$.

Denote

$$\alpha := \alpha(\Pi^{(i-1)}), \quad \beta := \beta(\Pi^{(i-1)}), \quad \gamma := \gamma(\Pi^{(i-1)}), \quad \delta := \delta(\Pi^{(i-1)}). \tag{17}$$

By Theorem 4,

$$\mathrm{pr}(\alpha^{-1}; \Pi) \in \mathscr{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j + e_r\right) \tag{18}$$

where $r = \mathrm{rank}(\Phi)$. Then, in view of (S_0), the path $\mathrm{pr}(\alpha^{-1}; \Pi) = \mathrm{pr}(\alpha; \Pi)^{-1}$ does not contain a boundary cycle of Π. By Lemma 7(d), (f) and Lemma 26(b):

15°. There is a p.o.b. cycle of Π of the form $\mathrm{pr}(\alpha; \Pi)^{-1}\omega_1\omega_2\omega_3$, where

(α) the path $\omega_1(\omega_2, \omega_3)$, if non-trivial, is a subpath of $\mathrm{pr}(\gamma^{-1}; \Pi)$ (of $\mathrm{pr}(\beta; \Pi)$, of $\mathrm{pr}(\delta; \Pi)$);

(β) if $\gamma(\delta)$ is trivial, then $\omega_1(\omega_3)$ is trivial (see Fig. 105).

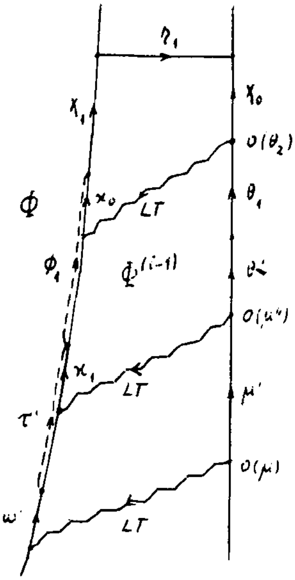


Fig. 104.

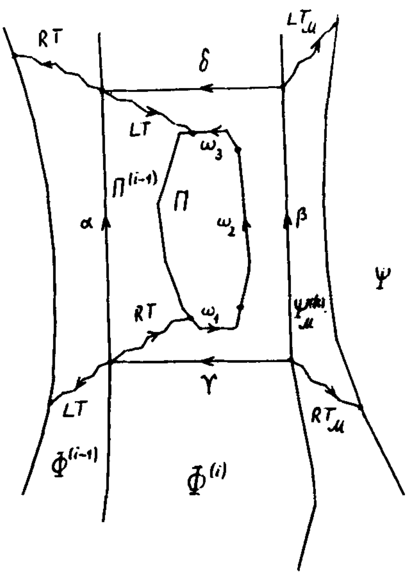


Fig. 105.

By Lemma 22(a), 15° and Theorem 4,

$$\omega_1, \omega_3 \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i} 13^{i-j} e_j\right). \tag{19}$$

Applying 2° to $\Pi^{(i-1)}$ and $\beta = \mu_{j_3}$ in place of $\Gamma^{(i-1)}$ and σ, and using 15°(α), we obtain $\omega_2 \in \mathcal{H}(\Pi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j + e_s)$ and therefore

$$\omega_1\omega_2\omega_3 \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 3\cdot 13^{i-j} e_j + 2e_i + e_s\right). \tag{20}$$

Since $\mathrm{pr}(\alpha;\Pi)^{-1}\omega_1\omega_2\omega_3$ is a p.o.b.c. of Π, it follows from (20) and (S_0) that $\mathrm{pr}(\alpha;\Pi)^{-1} \not\in \mathcal{H}(\Pi; \Sigma_{j=1}^{i} 4\cdot 13^{i-j} e_j)$. Hence

$$\mathrm{pr}(\alpha;\Pi) \not\in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right). \tag{21}$$

On the other hand, in view of (18) and (19),

$$\omega_2 \not\in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right). \tag{22}$$

We apply Theorem 4, with $i, \Phi, \Psi, \mu, \omega', \tau, \omega''$ replaced by $i-1, \Pi, \Phi, \alpha^{-1}$, $\mathrm{o}(\mathrm{pr}(\alpha^{-1};\Pi))$, $\mathrm{pr}(\alpha^{-1};\Pi)$, $\mathrm{t}(\mathrm{pr}(\alpha^{-1};\Pi))$. In view of (21), we conclude that there is a simple path $\xi_1 \in \mathrm{Br}_{\mathcal{M}}(i-1)$, connecting a vertex of $\mathrm{pr}(\alpha^{-1};\Pi)$ to a vertex of $\mathrm{pr}(\alpha^{-1};\Phi)$ and having the following properties:

16°. ξ_1 is a path in $\mathrm{S}(\alpha^{-1};\Pi)$ and $\mathrm{t}(\xi_1)$ is a vertex on the common boundary of Φ and $\Phi^{(i-1)}$.

17°. Let $\iota_1(\iota_2, \iota_3)$ be the (minimal) tail of $\mathrm{pr}(\alpha^{-1};\Pi)$ (of $\mathrm{pr}(\alpha^{-1};\Phi)$, of α^{-1}) such that $\mathrm{o}(\iota_1) = \mathrm{o}(\xi_1)$ ($\mathrm{o}(\iota_2) = \mathrm{t}(\xi_1)$, $\mathrm{o}(\iota_3) = \mathrm{t}(\xi_1)$). Then

(α) $\iota_1 \in \mathcal{H}(\Pi; \Sigma_{j=1}^{i-1} \frac{1}{2}\cdot 13^{i-j} e_j)$;

(β) $\iota_2 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} \frac{1}{2}\cdot 13^{i-j} e_j)$;

(γ) $\iota_2 \sim_{i-1} \iota_3 \mathrm{LT}(\mathrm{o}(\alpha);\Phi)$ (see Fig. 106).

By (22), ω_2 is non-trivial and then, by 15°(α), ω_2 is a subpath of $\mathrm{pr}(\beta;\Pi)$. Hence there exist paths κ', κ'' such that

$$\mathrm{pr}(\beta;\Pi) = \kappa'\omega_2\kappa''.$$

We now apply the induction hypothesis with $i, \Phi, \Psi, \mu, \omega', \tau, \omega''$ replaced by $i-1, \Pi, \Psi, \beta, \kappa', \omega_2, \kappa''$. In view of (22), we see that there is a simple path $\xi_2 \in \mathrm{Br}_{\mathcal{M}}(i-1)$, connecting a vertex of ω_2 to a vertex of $\mathrm{pr}_{\mathcal{M}}(\beta;\Psi)$ and having the following properties:

18°. ξ_2 is a path in $\mathrm{S}(\beta;\Pi)$ and $\mathrm{t}(\xi_2)$ is a common vertex of β and $\mathrm{pr}_{\mathcal{M}}(\beta;\Psi)$. (Here we are using (A(γ)) and the fact that $\mathrm{rank}(\Pi) = i < s = \mathrm{rank}(\Psi)$.)

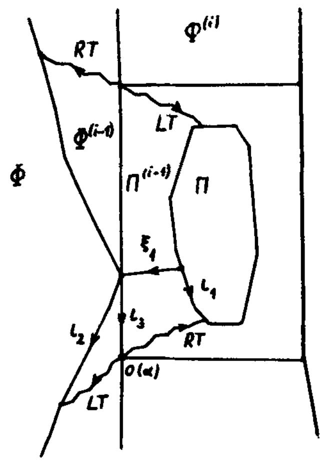


Fig. 106.

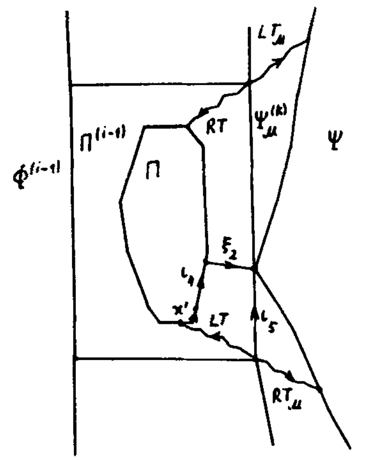


Fig. 107.

19°. Let ι_4 be the (minimal) head of $\mathrm{pr}(\beta;\Pi)$ such that $t(\iota_4)=o(\xi_2)$. Then $\iota_4\in\mathscr{H}(\Pi;\sum_{j=1}^{i-1}\frac{1}{2}\cdot 13^{i-j}e_j)$ (see Fig. 107).

Let η be the path obtained from $\xi_1^{-1}\iota_1\omega_1\iota_4\xi_2$ by deleting all its closed subpaths (if there are any). (See Fig. 108.)

We can now prove (C).

Indeed, by (19), 17°(α) and 19°, $\iota_1\omega_1\iota_4\in\mathscr{H}(\Pi;\sum_{j=1}^{i-1}2\cdot 13^{i-j}e_j+e_i)$. Since ξ_1 and ξ_2 belong to $\mathrm{Br}_{\mathcal{M}}(i-1)$, it follows from Lemma 1(a) and Definition 9 that $\xi_1^{-1}\iota_1\omega_1\iota_4\xi_2\in\mathrm{Br}_{\mathcal{M}}(i)$ and then, by Lemma 2, $\eta\in\mathrm{Br}_{\mathcal{M}}(i)$ (recall that $\mathcal{N}$ satisfies (SC_0)).

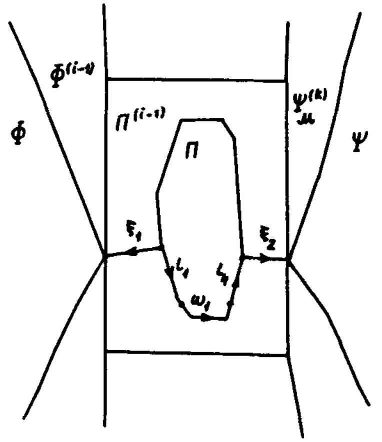


Fig. 108.

By (10) and (13), ϕ_3 is a subpath of τ and by (8) and (9), $\mu_{j_3} = \theta_3 = \beta$ is a subpath of μ; thus $\mathrm{pr}_{\mathcal{M}}(\beta;\Psi)$ is a subpath of $\mathrm{pr}_{\mathcal{M}}(\mu;\Psi)$. By the construction of ξ_1, $\mathrm{t}(\xi_1)$ is a vertex of $\mathrm{pr}(\alpha;\Phi)$. By 14°, $\mathrm{pr}(\alpha;\Phi) = \phi_3$. By the construction of ξ_2, $\mathrm{t}(\xi_2)$ is a vertex of $\mathrm{pr}(\beta;\Psi)$. We have $\mathrm{o}(\eta) = \mathrm{t}(\xi_1)$ and $\mathrm{t}(\eta) = \mathrm{t}(\xi_2)$. Therefore, η connects a vertex of τ to a vertex of $\mathrm{pr}_{\mathcal{M}}(\mu;\Psi)$. By construction, η is a simple path.

Using (10), (13), 14° and 17°, we see that the path τ_1 defined by

$$\tau_1 := \tau'\phi_1\phi_2\iota_2^{-1} \tag{23}$$

is a head of τ such that $\mathrm{t}(\tau_1) = \mathrm{o}(\iota_2) = \mathrm{t}(\xi_1) = \mathrm{o}(\eta)$. By 9°, 11°, 17°(β) and the assumption of Case 3, we have

$$\tau_1 = \tau'\phi_1\phi_2\iota_2^{-1} \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 4\frac{1}{2}\cdot 13^{i-j}e_j + 3e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

Since $\mathcal{M}$ satisfies (S_0), τ_1 cannot contain a boundary cycle of Φ and therefore τ_1 is the minimal head of τ such that $\mathrm{t}(\tau_1) = \mathrm{o}(\eta)$.

We now show that condition (A) is satisfied.

Take $\eta_1 := \eta$, $\eta_2 := t(\eta)$. Then, by 18°, $\mathrm{t}(\eta_1) = \mathrm{o}(\eta_2) = \mathrm{t}(\eta) = \mathrm{t}(\xi_2)$ is a vertex of β, hence of μ. By 16°, ξ_1 is a path in $\mathrm{S}(\alpha;\Pi)$ and, by 18°, ξ_2 is a path in $\mathrm{S}(\beta;\Pi)$. By 15°, 17° and 19°, $\iota_1\omega_1\iota_4$ is a boundary path of Π. Therefore, $\xi_1^{-1}\iota_1\omega_1\iota_4\xi_2$ is contained in $\mathrm{clos}(\Pi^{(i-1)})$; then η is also contained in $\mathrm{clos}(\Pi^{(i-1)})$. By Definitions 20, 27 and 32, $\mathrm{clos}(\Pi^{(i-1)}) \subseteq \mathrm{supp}(\mathrm{S}(\beta;\Phi)) \subseteq \mathrm{supp}(\mathrm{S}(\mu;\Phi))$ and so $\eta = \eta_1$ is a path in $\mathrm{S}(\mu;\Phi)$. By 18°, the (trivial) path $\eta_2 = \mathrm{t}(\eta) = \mathrm{t}(\xi_2)$ is a vertex of $\mathrm{pr}(\beta;\Psi)$, hence of $\mathrm{pr}(\mu;\Psi)$. Then, of course, η_2 is a path in $\mathrm{S}(\mu;\Phi)$. We have verified (A(α)).

Let ι_5 be the head of $\beta = \theta_3 = \mu_{j_3}$ such that $\mathrm{t}(\iota_5) = \mathrm{t}(\eta) = \mathrm{t}(\eta_1) = \mathrm{t}(\xi_2)$ (see Fig. 107). In view of (8) and (12), the path μ_0 defined by

$$\mu_0 := \mu'\theta'\theta_1\theta_2\iota_5 \tag{24}$$

is the head of μ such that $\mathrm{t}(\mu_0) = \mathrm{t}(\eta_1)$.

Using 7°(β), Lemma 28(d), 17°(γ), (23), (24), Lemma 15, Lemma 26 and reasoning exactly as in the proof of Theorem 4, we obtain $\omega'\tau_1 \sim_i \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu_0\eta_1^{-1}$ (see Fig. 109). We have thus verified (A(β)). (A(γ)) is also satisfied, as $\eta_2 = \mathrm{t}(\eta)$ is trivial.

This proves (C) in Case 3. Since Cases 1, 2, 3 exhaust all possibilities, (C) is proved in its entirety.

Similarly, one can prove:

(C′) Either $\tau \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j}e_j)$, or there is a simple path $\eta' \in \mathrm{Br}_{\mathcal{M}}(i)$,

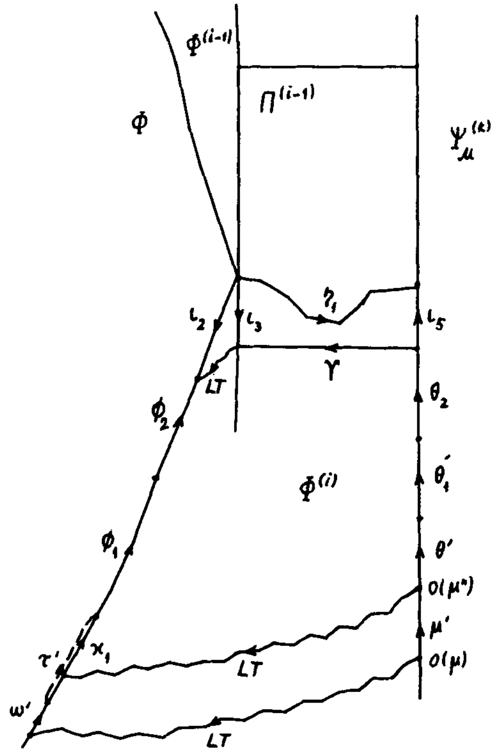


Fig. 109.

connecting a vertex of τ to a vertex of $\mathrm{pr}_{\mathcal{M}}(\mu;\Psi)$, having property (A′) and such that, if τ_2 is the (minimal) tail of τ satisfying $o(\tau_2)=o(\eta')$, then

$$\tau_2 \in \mathcal{H}\left(\Phi;\ \sum_{j=1}^{i}\frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

We can now deduce the remaining assertions of Theorem 5 from (C) and (C′). As in the proof of Theorem 4, assuming that $\tau \notin \mathcal{H}(\Phi;\Sigma_{j=1}^{i}13^{i+1-j}e_j)$, we see that, for some subpath θ of τ, $\tau=\tau_1\theta\tau_2$. Next, letting ξ_0 denote the path obtained by reducing $\mu_0^{-1}\mu\mu_0'^{-1}$ and noting (A(β)) and (A′(β)), we obtain $\theta \sim_i \eta_1\xi_0\eta_1'^{-1}$ (see Fig. 97). Now, by (A(α)) and (A′(α)), the paths η_2 and η_2' are in $S(\mu;\Psi)$; hence we conclude that there is a boundary path ξ of Ψ such that $\xi_0 \sim_i \eta_2\xi\eta_2^{-1}$ (see Fig. 98). Then

$$\theta \underset{i}{\sim} \eta_1\xi_0\eta_1'^{-1} \underset{i}{\sim} \eta_1\eta_2\xi\eta_2'^{-1}\eta_1'^{-1} = \eta\xi\eta'^{-1}.$$

Since η, η' are simple paths belonging to $\mathrm{Br}_{\mathcal{M}}(i)$, it follows from Definition 9 that $\theta \in \mathcal{P}_{\mathcal{M}}(\Phi;s)=\mathcal{I}_{\mathcal{M}}(\Phi;e_s)$.

In view of (5) and Definition 9, we have

$$\tau = \tau_1 \theta \tau_2 \in \mathcal{I}_{\mathcal{M}}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j + e_s\right).$$

We have shown that, if τ is an arbitrary subpath of $\mathrm{pr}(\mu;\Phi)$, then either $\tau \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j})$ or $\tau \in \mathcal{I}_{\mathcal{M}}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j} e_j + e_s)$. Hence, by Definition 9,

$$\mathrm{pr}(\mu;\Phi) \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j + e_s\right).$$

This completes the proof of the theorem.

We need also the case when $\mathrm{rank}(\Psi) \leqq i$. This case is much simpler than the case when $\mathrm{rank}(\Psi) > i$.

THEOREM 6. *Under the conditions of Theorem* 5, *let us assume that* $k < \mathrm{rank}(\Psi) = s \leqq i$. *Then*

$$\mathrm{pr}(\mu;\Phi) \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j\right).$$

PROOF. We proceed by induction on $i - k$.

If $i - k = 0$, the statement of the theorem is vacuous, so we assume that $i - k > 0$.

1°. Let Γ be a region of $\mathcal{N}$ of rank i such that $\Gamma^{(i-1)} \in \mathcal{L}_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})$. Let σ be a subpath of μ which is a boundary path of $\Gamma^{(i-1)}$. Then $\mathrm{pr}(\sigma;\Gamma) \in \mathcal{H}(\Gamma; \Sigma_{j=1}^{i} 13^{i-j} e_j)$.

Indeed, as a subpath of μ, σ is a n.o.b.p. of $\Psi_{\mathcal{M}}^{(k)}$. If $\mathrm{rank}(\Psi) = s \leqq i - 1$ then, by the induction hypothesis,

$$\mathrm{pr}(\sigma;\Gamma) \in \mathcal{H}\left(\Gamma; \sum_{j=1}^{i-1} 13^{i-j} e_j\right) \subseteq \mathcal{H}\left(\Gamma; \sum_{j=1}^{i} 13^{i-j} e_j\right).$$

If $s = i$ then, using Theorem 5, we obtain

$$\mathrm{pr}(\sigma;\Gamma) \in \mathcal{H}\left(\Gamma; \sum_{j=1}^{i-1} 13^{i-j} e_j + e_i\right) = \mathcal{H}\left(\Gamma; \sum_{j=1}^{i} 13^{i-j} e_j\right),$$

as required.

Now we have:

2°. Under the conditions of 1°, $\sigma \neq \beta(\Gamma^{(i-1)})$.

Let $\alpha := \alpha(\Gamma^{(i-1)})$, $\beta := \beta(\Gamma^{(i-1)})$, $\gamma := \gamma(\Gamma^{(i-1)})$, $\delta := \delta(\Gamma^{(i-1)})$. Reasoning as in 3° of Theorem 5, we can find a boundary cycle $\nu_1\nu_2$ of Γ such that ν_1 is a subpath of $\mathrm{pr}(\beta;\Gamma)$ and ν_2 is a subpath of $\mathrm{pr}(\delta\alpha^{-1}\gamma^{-1};\Gamma)$.

If $\sigma = \beta = \beta(\Gamma^{(i-1)})$, then, by 1°, $\nu_1 \in \mathcal{H}(\Gamma; \Sigma_{j=1}^{i} 13^{i-j} e_j)$.

If $d_{N^{(i-1)}}(\Gamma^{(i-1)}, \Phi^{(i-1)}) > 1$ then, as in 3° of Theorem 5, we obtain $\nu_2 \in \mathcal{H}(\Gamma; \Sigma_{j=1}^{i} 4\cdot 13^{i-j} e_j)$ and then

$$\nu_1 \nu_2 \in \mathcal{H}\left(\Gamma; \sum_{j=1}^{i} 5\cdot 13^{i-j} e_j\right),$$

contradicting (S_0). If $d_{N^{(i-1)}}(\Gamma^{(i-1)}, \Phi^{(i-1)}) = 1$ then, by Lemma 22(a) (b) and Theorem 4 we obtain $\nu_2 \in \mathcal{H}(\Gamma; \Sigma_{j=1}^{i-1} 3\cdot 13^{i-j} e_j + 2e_i + e_r)$ and then

$$\nu_1 \nu_2 \in \mathcal{H}\left(\Gamma; \sum_{j=1}^{i-1} 4\cdot 13^{i-j} e_j + 3e_i + e_r\right),$$

also contradicting (S_0). Thus, $\sigma \neq \beta(\Gamma^{(i-1)})$, as required.

We now apply Proposition 1 to the path μ, with M, Φ, Φ' replaced by $N^{(i-1)}$, $\Phi^{(i-1)}$ and $\Phi^{(i)}$. There results a factorization $\mu = \mu' \mu'' \mu'''$ and, if μ'' is non-trivial, a further factorization $\mu'' = \mu_1 \mu_2 \cdots \mu_h$ such that

3°. μ' is a head of $\mathrm{RT}(\mathrm{o}(\mu); \Phi^{(i-1)})$.

4°. μ'''^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\mu); \Phi^{(i-1)})$.

5°. If μ'' is non-trivial then

(α) μ'' is on the boundary of $(\Phi^{(i-1)})^1$;

(β) the factorization $\mu'' = \mu_1 \mu_2 \cdots \mu_h$ is the l.h.s. factorization of μ'' in $N^{(i-1)}$;

(γ) for any j, $1 \leqq j \leqq h$, if μ_j is not on the boundary of $\Phi^{(i-1)}$, then $\mu_j = \beta(\Pi_j^{(i-1)})$ for some $\Pi_j^{(i-1)} \in \mathcal{L}^1_{N^{(i-1)}}(\Phi^{(i-1)})$.

Comparing 5°(γ) with 2°, we obtain

6°. If μ'' is non-trivial, it is on the boundary of $\Phi^{(i-1)}$.

Let τ be a subpath of $\mathrm{pr}(\mu; \Phi)$. As in the proof of Theorems 4, 5, there is a factorization $\tau = \tau' \tau'' \tau'''$ with the following properties:

7°. $\tau'(\tau'', \tau''')$ is either trivial or a subpath of $\mathrm{pr}(\mu'; \Phi)$ (of $\mathrm{pr}(\mu''; \Phi)$, of $\mathrm{pr}(\mu'''; \Phi)$).

8°. If μ'' is trivial then τ'' is trivial.

As in the proof of Theorems 4, 5, we have:

9°. $\tau', \tau''' \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 2\cdot 13^{i-j} e_j)$.

If τ'' is trivial, then $\tau = \tau' \tau''' \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 4\cdot 13^{i-j} e_j) \subseteq \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j} e_j)$. If τ'' is non-trivial then, by 8°, μ'' is also non-trivial. Then, by 6°, μ'' is a p.o.b.p. of $\Phi^{(i-1)}$ which is also a n.o.b.p. of $\Psi^{(k)}_{\mathcal{M}}$.

If $\mathrm{rank}(\Psi) = s \leqq i-1$ then, by the induction hypothesis, $\mathrm{pr}(\mu''; \Phi) \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$ and then, by 7°, we have also

$$\tau'' \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i-j} e_j\right).$$

If $\operatorname{rank}(\Psi) = s = i$ then, by Theorem 5,

$$\tau'' \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 13^{i-j}e_j + e_i\right) = \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i-j}e_j\right).$$

Then, by 9°,

$$\tau = \tau'\tau''\tau''' \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 5\cdot 13^{i-j}e_j\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j\right).$$

We have shown that any subpath of $\operatorname{pr}(\mu;\Phi)$ belongs to $\mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j}e_j)$. In particular,

$$\operatorname{pr}(\mu;\Phi) \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j\right).$$

The theorem is proved.

7.2. In this section we consider a somewhat different situation than in Theorem 4. Instead of looking at a path on the common boundary of two regions in $M^{(i)}$, we consider a boundary path μ of a region in $M^{(i)}$ which belongs to a special class of paths which we now define.

Definition 34. *The sets of paths* $\mathcal{G}^i(c)$. Let $i \geqq 1$ and $c = \Sigma_{j\geqq 1} c_j e_j$. We say that a path μ in M belongs to $\mathcal{G}^i(c)$ if and only if given:

(α) a factorization $\mu = \mu_1\mu_2\mu_3$;

(β) simple paths $\sigma, \tau \in \operatorname{Br}(i-1)$;

(γ) a boundary path ν_1 of a region Φ in M, of rank i, such that

(δ) $\mu_2 \sim_{i-1} \sigma^{-1}\nu_1\tau$ (see Fig. 110)

we have the following:

(1) ν_1 does not contain a boundary cycle of Φ;

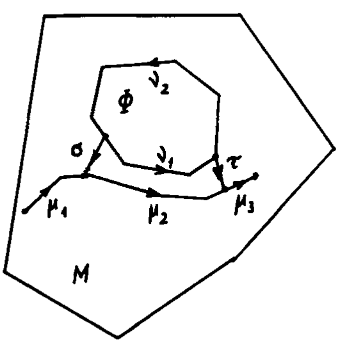


Fig. 110.

(2) if ν_2 is a boundary path of Φ such that $\nu_1\nu_2$ is a boundary cycle of Φ then $\nu_2 \not\in \mathcal{H}(\Phi; c)$.

THEOREM 7. *Let $\mathcal{M} = (M, \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}, <)$ be an ordered n-ranked map satisfying condition* (S_0) *and condition* (SC_i) *for some $i, 0 \leqq i < n$. Let Φ be a region in M, of rank $r > i$, and $\Phi^{(i)}$ the corresponding region in $M^{(i)}$. Let μ be a p.o.b.p. of $\Phi^{(i)}$ such that*

(1) $$\mu \in \bigcap_{h=1}^{i} \mathcal{G}^h\left(\sum_{j=1}^{h-1} 5 \cdot 13^{h-j} e_j + 4e_h\right).$$

Assume, given a factorization

(2) $$\operatorname{pr}(\mu; \Phi) = \omega' \tau \omega'',$$

then either

(3) $$\tau \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j} e_j\right)$$

or there exist two simple paths $\eta, \eta' \in \operatorname{Br}(i)$ in $S(\mu; \Phi)$, each connecting a vertex of τ to a vertex of μ, with the following properties (see Fig. 111):

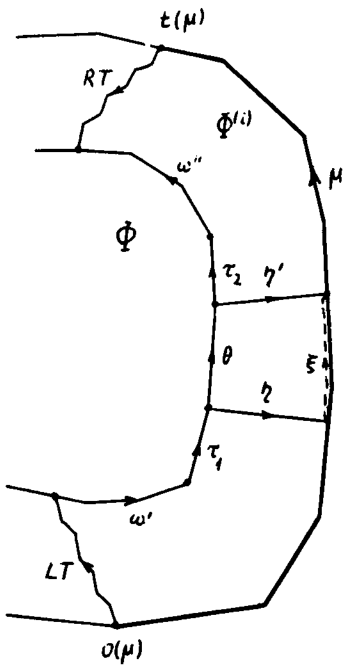


Fig. 111.

(a) *Let* τ_1 *be the minimal head of* τ *such that* $\mathrm{t}(\tau_1)=\mathrm{o}(\eta)$. *Then* $\tau_1 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j)$.

(a') *Let* τ_2 *be the minimal tail of* τ *such that* $\mathrm{o}(\tau_2)=\mathrm{o}(\eta')$. *Then* $\tau_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j)$.

(b) *There is a head* μ_0 *of* μ, *such that* $\mathrm{t}(\mu_0)=\mathrm{t}(\eta)$, *for which* $\omega'\tau_1 \sim_i \mathrm{LT}(\mathrm{o}(\mu);\Phi)^{-1}\mu_0\eta^{-1}$.

(b') *There is a tail* μ_0' *of* μ, *such that* $\mathrm{o}(\mu_0')=\mathrm{t}(\eta')$, *for which*

$$\tau_2\omega'' \underset{i}{\sim} \eta'\mu_0'\mathrm{RT}(\mathrm{t}(\mu);\Phi).$$

(c) $\tau=\tau_1\theta\tau_2$ *for some subpath* θ *of* τ. *Furthermore, for some subpath* ξ *of* μ *or* μ^{-1}, *connecting* $t(\eta)$ *to* $t(\eta')$,

$$\theta \underset{i}{\sim} \eta\xi\eta'^{-1}.$$

COROLLARY. *Under the assumptions of Theorem* 7, *assume in addition that* $\mu \in \mathscr{G}'(\Sigma_{j\geqq 1} c_je_j)$ *for some* $c_j \geqq 0$, *and that, for some boundary path* ω *of* Φ, $\tau\omega$ *is a b.c. of* Φ. *Then either* (3) *holds or*

$$\omega \notin \mathscr{H}\left(\Phi; \sum_{j\geqq 1} c_je_j - \sum_{j=1}^{i} 13^{i+1-j}e_j\right). \tag{4}$$

PROOF. Let us assume that neither (3) nor (4) is true. Then, by (a) and (a'),

$$\tau_2\omega\tau_1 \in \mathscr{H}\left(\Phi; \sum_{j\geqq 1} c_je_j\right).$$

By Lemma 1(a), (c), η^{-1} and η'^{-1} belong to $\mathrm{Br}(i) \subseteq \mathrm{Br}(r-1)$. Since $i \leqq r-1$, we have also $\xi \sim_{r-1} \eta^{-1}\theta\eta'$ and then, by Definition 34, any path in M, that contains ξ or ξ^{-1} as a subpath, cannot belong to $\mathscr{G}'(\Sigma_{j\geqq 1} c_je_j)$. In view of (c), this contradicts our assumption. (See Fig. 112.)

PROOF OF THEOREM 7. We proceed by induction on i.

If $i=0$, then $\mu=\mathrm{pr}(\mu;\Phi)=\omega'\tau\omega''$ (see Fig. 113). Take $\eta:=\mathrm{o}(\tau)$, $\eta':=\mathrm{t}(\tau)$. Then

$$\tau_1=\mathrm{o}(\tau), \quad \theta=\tau, \quad \tau_2=\mathrm{t}(\tau), \quad \mu_0=\omega', \quad \mu_0'=\omega'', \quad \xi=\theta$$

and then conditions (a), (a'), (b), (b'), (c) are obviously satisfied.

Assume now that $i>0$.

We begin with the following statement.

1°. Let Γ be a region in M, of rank i, such that $\Gamma^{(i-1)} \in \mathscr{L}^k_{\mathscr{X}^{(i-1)}}(\Phi^{(i-1)})$ with $k>1$. Let σ be a boundary path of $\Gamma^{(i-1)}$ which is a subpath of μ. Then $\sigma \neq \beta(\Gamma^{(i-1)})$.

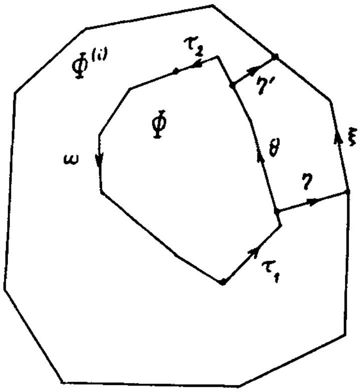


Fig. 112.

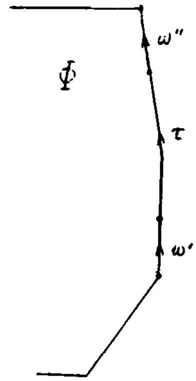


Fig. 113.

Indeed, as in 3° of Theorem 5, we can find a boundary cycle $\nu_1\nu_2$ of Γ such that ν_1 is a subpath of $\mathrm{pr}(\beta(\Gamma^{(i-1)});\Gamma)$ and $\nu_2 \in \mathscr{H}(\Gamma;\Sigma_{j=1}^{i} 4\cdot 13^{i-j}e_j)$.

By our assumption, $\mu \in \mathscr{G}^i(\Sigma_{j=1}^{i-1} 5\cdot 13^{i-j}e_j + 4e_i)$, therefore, by Definition 34, σ also belongs to $\mathscr{G}^i(\Sigma_{j=1}^{i-1} 5\cdot 13^{i-j}e_j + 4e_i)$. If $\sigma = \beta(\Gamma^{(i-1)})$, then applying the induction hypothesis and the corollary of Theorem 7, we obtain that either $\nu_1 \in \mathscr{H}(\Gamma;\Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ or $\nu_2 \notin \mathscr{H}(\Gamma;\Sigma_{j=1}^{i} 4\cdot 13^{i-j}e_j)$. The second statement is impossible and the first statement implies

$$\nu_1\nu_2 \in \mathscr{H}\left(\Gamma;\sum_{j=1}^{i-1} 5\cdot 13^{i-j}e_j + 4e_i\right)$$

contradicting (S_0). Therefore, $\sigma \neq \beta(\Gamma^{(i-1)})$, as required.

The rest of the proof is completely similar to the proof of Theorem 5.

We prove the following statement:

(C) Either $\tau \in \mathscr{H}(\Phi;\Sigma_{j=1}^{i} 13^{i+1-j}e_j)$ or there exists a simple path $\eta \in \mathrm{Br}(i)$ in $S(\mu;\Phi)$ connecting a vertex of τ to a vertex of μ and having properties (a), (b).

Applying Proposition 1 with M, Φ, Φ' replaced by $M^{(i-1)}$, $\Phi^{(i-1)}$ and $\Phi^{(i)}$, we obtain a factorization

$$\mu = \mu'\mu''\mu''' \tag{5}$$

and, if μ'' is non-trivial, a further factorization

$$\mu'' = \mu_1\mu_2\cdots\mu_h \tag{6}$$

such that

2°. μ' is a head of $\mathrm{RT}(\mathrm{o}(\mu);\Phi^{(i-1)})$.

3°. μ'''^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\mu);\Phi^{(i-1)})$.

4°. If μ'' is non-trivial then

(α) μ'' is on the boundary of $(\Phi^{(i-1)})^1$;

(β) the factorization (6) is the l.h.s. factorization of μ'' in $M^{(i-1)}$;

(γ) for any $j, 1 \leqq j \leqq h$, if μ_j is not on the boundary of $\Phi^{(i-1)}$, then $\mu_j = \beta(\Pi_j^{(i-1)})$ for some $\Pi_j^{(i-1)} \in \mathcal{L}^1_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$.

As in the proof of Theorems 4, 5 we conclude that there is a factorization

$$\tau = \tau'\tau''\tau''' \tag{7}$$

with the following properties:

5°. If $\tau'(\tau'', \tau''')$ is non-trivial, it is a subpath of $\mathrm{pr}(\mu'; \Phi)$ (of $\mathrm{pr}(\mu''; \Phi)$, of $\mathrm{pr}(\mu'''; \Phi)$). Moreover, there are paths κ_1, κ_2 such that

(α) $\mathrm{pr}(\mu''; \phi) = \kappa_1\tau''\kappa_2$;

(β) $\mathrm{lpr}(\mu'; \Phi)\kappa_1 = \omega'\tau'$;

(γ) $\kappa_2 \mathrm{rpr}(\mu'''; \Phi) = \tau'''\omega''$ (see Fig. 76).

6°. If μ'' is trivial then τ'' is trivial.

As in the proof of Theorems 4, 5 we obtain

7°. $\tau' \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 2\cdot 13^{i-j}e_j)$ and $\tau''' \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 2\cdot 13^{i-j}e_j)$.

Using (7), we have

8°. If τ'' is trivial then $\tau = \tau'\tau''' \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 4\cdot 13^{i-j}e_j) \subseteq \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j}e_j)$.

In what follows, we assume that τ'' is non-trivial; then, by 6°, μ'' is also non-trivial.

Let S be the subset of $\{\mu_1, \mu_2, \cdots, \mu_h\}$ defined as follows:

$$S := \{\mu_j \mid \mu_j \text{ is on the boundary of } \Phi^{(i-1)}\}. \tag{8}$$

Using Lemma 17(a) and 4°(β), we obtain that the paths μ_{j-1}, μ_j cannot both belong to S. We apply Lemma 28 with μ, ν replaced by μ'', τ''. There result factorizations

$$\mu'' = \theta'\theta_1\theta_2\theta_3\theta'' \tag{9}$$

and

$$\tau'' = \phi_1\phi_2\phi_3\psi \tag{10}$$

with the properties described in Lemma 28.

As in the proof of Theorems 4, 5 one shows that

9°. $\phi_1, \phi_3 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i-j}e_j)$.

Here we use the reference to 4°(γ).

We have the following possibilities:

(1) $\phi_2 \notin \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$;

(2) $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$ and ψ is trivial;

(3) $\phi_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$ and ψ is non-trivial.

We consider each of these cases separately.

Case 1. $\phi_2 \notin \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$.

In this case ϕ_2 is non-trivial. Hence, by Lemma 28(c), θ_2 is non-trivial, and then

$$\theta_2 = \mu_{j_2} \in S. \tag{11}$$

By (8), μ_{j_2} is on the boundary of $\Phi^{(i-1)}$. By Lemma 28(c), there are paths κ_0, κ_0' for which $\mathrm{pr}(\mu_{j_2}; \Phi) = \kappa_0 \phi_2 \kappa_0'$ (see Fig. 114).

We apply the induction hypothesis with $i, \mu, \omega', \tau, \omega''$ replaced by $i-1$, $\mu_{j_2}, \kappa_0, \phi_2, \kappa_0'$. Since $\phi_2 \notin \mathscr{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j} e_j)$, it follows that there is a simple path $\eta \in \mathrm{Br}(i-1)$ in $\mathrm{S}(\mu_{j_2}; \Phi)$, connecting a vertex of ϕ_2 to a vertex of μ_{j_2} and having the following properties:

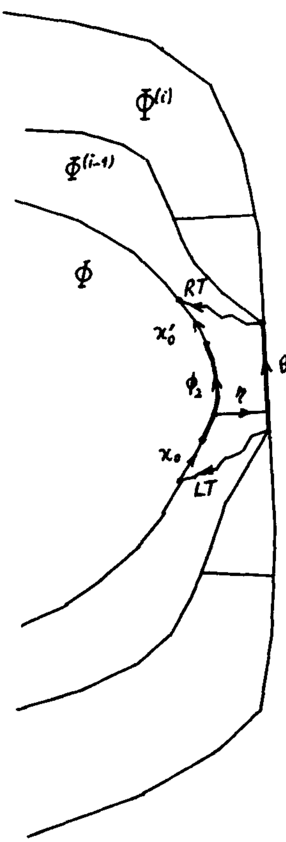


Fig. 114.

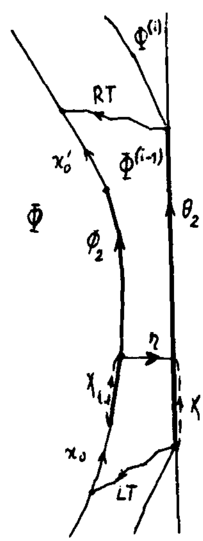


Fig. 115.

10°. Let χ_1 be the minimal head of ϕ_2 such that $t(\chi_1) = o(\eta)$. Then

$$\chi_1 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} \frac{1}{2} \cdot 13^{i-j} e_j\right).$$

11°. For some head χ_0 of μ_{j_2} such that $t(\chi_0) = t(\eta)$,

$$\kappa_0 \chi_1 \underset{i-1}{\sim} \mathrm{LT}(o(\mu_{j_2}); \Phi)^{-1} \chi_0 \eta^{-1} \qquad \text{(see Fig. 115)}.$$

We now prove (C).

By (7) and (10), ϕ_2 is a subpath of τ. By (5) and (6), μ_{j_2} is a subpath of μ. By Lemma 1(c), $\mathrm{Br}(i-1) \subseteq \mathrm{Br}(i)$. Therefore, η is a simple path in $S(\mu; \Phi)$ belonging to $\mathrm{Br}(i)$ and connecting a vertex of τ to a vertex of μ. Define

$$\tau_1 := \tau' \phi_1 \chi_1. \tag{12}$$

Then by (7), (10) and 10°, τ_1 is a head of τ such that $t(\tau_1) = o(\eta)$. By 7°, 9° and 10°,

$$\tau_1 = \tau' \phi_1 \chi_1 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 3\frac{1}{2} \cdot 13^{i-j} e_j + 3e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2} \cdot 13^{i+1-j} e_j\right).$$

Since $\mathcal{M}$ satisfies (S_0), τ_1 cannot contain a boundary cycle of Φ and therefore τ_1 is the minimal head of τ such that $t(\tau_1) = o(\eta)$. We have verified (a). Take

$$\mu_0 := \mu' \theta' \theta_1 \chi_0. \tag{13}$$

By (5), (9) and 11°, μ_0 is a head of μ such that $t(\mu_0) = o(\eta)$. We have the situation in Fig. 104.

Using 5°(β), Lemma 15 and Lemma 26, we obtain

$$\omega' \tau' \underset{i}{\sim} \mathrm{LT}(o(\mu); \Phi)^{-1} \mu' \mathrm{LT}(o(\mu''); \Phi) \kappa_1. \tag{14}$$

Lemma 28(c) gives

$$\kappa_1 \phi_1 \underset{i}{\sim} \mathrm{LT}(o(\mu''); \Phi)^{-1} \theta' \theta_1 \mathrm{LT}(o(\mu_{j_2}); \Phi) \kappa_0. \tag{15}$$

Using (12), (13), (14), (15) and 11°, we obtain

$$\omega' \tau_1 \sim \mathrm{LT}(o(\mu); \Phi)^{-1} \mu_0 \eta^{-1}.$$

We have verified (b) too. This completes the proof of (C) in Case 1.

Case 2. $\phi_2 \in \mathcal{H}(\Phi; \sum_{j=1}^{i-1} 13^{i-j} e_j)$ and ψ is trivial.

In this case, by (7) and (10), $\tau = \tau'\phi_1\phi_2\phi_3\tau'''$. Then, by 7° and 9°,

$$\tau = \tau'\phi_1\phi_2\phi_3\tau''' \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 7\cdot 13^{i-j}e_j + 6e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} 13^{i+1-j}e_j\right)$$

and therefore (C) is true.

Case 3. $\phi_2 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ and ψ is non-trivial.

Since ψ is non-trivial, it follows from Lemma 28 (a), (d), (c) that $\theta_3 = \mu_{j_3} \notin S$ and $\phi_3 = \mathrm{pr}(\mu_{j_3}; \Phi)$. By (8) and 4°(γ), there is a region $\Pi^{(i-1)} \in \mathcal{L}^1_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$ such that

(16) $$\theta_3 = \mu_{j_3} = \beta(\Pi^{(i-1)}).$$

Denote

(17) $$\alpha := \alpha(\Pi^{(i-1)}), \quad \beta := \beta(\Pi^{(i-1)}), \quad \gamma := \gamma(\Pi^{(i-1)}), \quad \delta := \delta(\Pi^{(i-1)}).$$

By Definitions 19, 26, 27 and 32,

12°. $\phi_3 = \mathrm{pr}(\beta; \Phi) = \mathrm{pr}(\alpha; \Phi)$.

By Theorem 4,

(18) $$\mathrm{pr}(\alpha^{-1}; \Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j}e_j + e_r\right)$$

where $r = \mathrm{rank}(\Phi)$. Then, in view of (S_0), the path $\mathrm{pr}(\alpha^{-1}; \Pi) = \mathrm{pr}(\alpha; \Pi)^{-1}$ does not contain a boundary cycle of Π. By Lemma 7(d), (f) and Lemma 26:

13°. There is a p.o.b.c. of Π of the form $\mathrm{pr}(\alpha; \Pi)^{-1}\omega_1\omega_2\omega_3$, where

(α) the path $\omega_1(\omega_2, \omega_3)$, if non-trivial, is a subpath of $\mathrm{pr}(\gamma^{-1}; \Pi)$ (of $\mathrm{pr}(\beta; \Pi)$, of $\mathrm{pr}(\delta; \Pi)$);

(β) if $\gamma(\delta)$ is trivial, then $\omega_1(\omega_3)$ is trivial (see Fig. 116).

Applying the induction hypothesis and the Corollary, with i, r, Φ, μ, τ, ω, replaced by $i-1$, i, Π, $\beta = \mu_{j_3}$, ω_2, $\omega_3\mathrm{pr}(\alpha^{-1}; \Pi)\omega_1$, we see that either $\omega_2 \in \mathcal{H}(\Pi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ or

(19) $$\omega_3\mathrm{pr}(\alpha^{-1}; \Pi)\omega_1 \notin \mathcal{H}\left(\Pi; \sum_{j=1}^{i} 4\cdot 13^{i-j}e_j\right).$$

In view of (S_0), if $\omega_2 \in \mathcal{H}(\Pi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$, then (19) also holds. Thus, (19) holds in each case. By Lemma 22(a), 13° and Theorem 4,

(20) $$\omega_1, \omega_3 \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i} 13^{i-j}e_j\right).$$

Comparing (19) and (20), we obtain $\mathrm{pr}(\alpha^{-1}; \Pi) \notin \mathcal{H}(\Pi; \Sigma_{j=1}^{i} 2\cdot 13^{i-j}e_j)$. Then, of course,

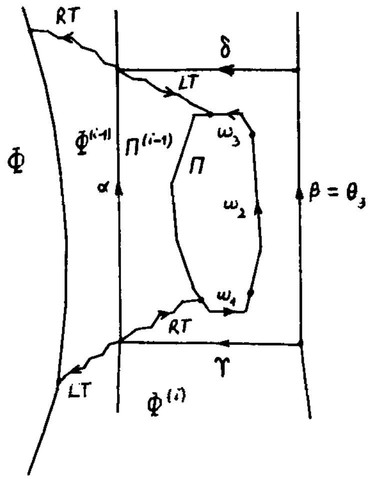


Fig. 116.

(21) $$\mathrm{pr}(\alpha^{-1};\Pi) \notin \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right).$$

On the other hand, by (18) and (20),

$$\omega_3 \mathrm{pr}(\alpha^{-1};\Pi)\omega_1 \in \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 3\cdot 13^{i-j} e_j + 2e_i + e_r\right),$$

and then, in view of (S_0), $\omega_2 \notin \mathcal{H}(\Pi; \Sigma_{j=1}^{i} 4\cdot 13^{i-j} e_j)$; hence

(22) $$\omega_2 \notin \mathcal{H}\left(\Pi; \sum_{j=1}^{i-1} 13^{i-j} e_j\right).$$

We apply Theorem 4, with $i, \Phi, \Psi, \mu, \omega', \tau', \omega''$ replaced by $i-1, \Pi, \Phi, \alpha^{-1}$, $\mathrm{o}(\mathrm{pr}(\alpha^{-1};\Pi))$, $\mathrm{pr}(\alpha^{-1};\Pi)$, $\mathrm{t}(\mathrm{pr}(\alpha^{-1};\Pi))$ (see Fig. 106). In view of (21), we conclude that there is a simple path $\xi_1 \in \mathrm{Br}(i-1)$, connecting a vertex of $\mathrm{pr}(\alpha^{-1};\Pi)$ to a vertex of $\mathrm{pr}(\alpha^{-1};\Phi)$ and having the following properties:

13°. ξ_1 is a path in $\mathrm{S}(\alpha^{-1};\Pi)$ and $\mathrm{t}(\xi_1)$ is a vertex on the common boundary of Φ and $\Phi^{(i-1)}$.

14°. Let $\iota_1(\iota_2, \iota_3)$ be the (minimal) tail of $\mathrm{pr}(\alpha^{-1};\Pi)$ (of $\mathrm{pr}(\alpha^{-1};\Phi)$, of α^{-1}) such that $\mathrm{o}(\iota_1) = \mathrm{o}(\xi_1)$ ($\mathrm{o}(\iota_2) = \mathrm{t}(\xi_1)$, $\mathrm{o}(\iota_3) = \mathrm{t}(\xi_1)$). Then

(α) $\iota_1 \in \mathcal{H}(\Pi; \Sigma_{j=1}^{i-1} \frac{1}{2}\cdot 13^{i-j} e_j)$;

(β) $\iota_2 \in \mathcal{H}(\Pi; \Sigma_{j=1}^{i-1} \frac{1}{2}\cdot 13^{i-j} e_j)$;

(γ) $\iota_2 \sim_{i-1} \iota_3 \mathrm{LT}(\mathrm{o}(\alpha);\Phi)$.

By (22), ω_2 is non-trivial and then, by 13°(α), ω_2 is a subpath of $\mathrm{pr}(\beta;\Pi)$. Hence there exist paths κ', κ'' such that $\mathrm{pr}(\beta;\Pi) = \kappa'\omega_2\kappa''$.

We now apply the induction hypothesis with $i, \Phi, \mu, \omega', \tau, \omega''$ replaced by $i-1, \Pi, \beta, \kappa', \omega_2, \kappa''$ (see Fig. 117). Here we use the fact that, by (5) and (9), $\beta = \mu_{j_3}$ is a subpath of μ; hence, by (1),

$$\beta \in \bigcap_{h=1}^{i-1} \mathscr{G}^h\left(\sum_{j=1}^{h-1} 5\cdot 13^{h-j}e_j + 4e_h\right).$$

In view of (22), there is a simple path $\xi_2 \in \mathrm{Br}(i-1)$ in $\mathrm{S}(\beta;\Pi)$ connecting a vertex of ω_2 to a vertex of β and such that, if ι_4 is the (minimal) head of ω_2 for which $\mathrm{o}(\xi_2) = \mathrm{o}(\iota_4)$ then

$$\iota_4 \in \mathscr{H}\left(\Pi; \sum_{j=1}^{i-1} \frac{1}{2}\cdot 13^{i-j}e_j\right). \tag{23}$$

Let η be the path obtained from $\xi_1^{-1}\iota_1\omega_1\iota_4\xi_2$ by deleting all its closed subpaths (if there are any).

We can now prove (C).

Indeed, by (20), (23) and 14°(α),

$$\iota_1\omega_1\iota_4 \in \mathscr{H}\left(\Pi; \sum_{j=1}^{i-1} 2\cdot 13^{i-j}e_j + e_i\right).$$

Since ξ_1 and ξ_2 belong to $\mathrm{Br}(i-1)$, it follows from Lemma 1(a) and Definition 9 that $\xi_1^{-1}\iota_1\omega_1\iota_4\xi_2 \in \mathrm{Br}(i)$ and then, since $\mathscr{M}$ satisfies (SC_0), by Lemma 2, we obtain $\eta \in \mathrm{Br}(i)$. By (7) and (10), ϕ_3 is a subpath of τ and by (5) and (6), $\mu_{j_3} = \beta$ is a subpath of μ. By the construction of ξ_1, $\mathrm{t}(\xi_1)$ is a vertex of $\mathrm{pr}(\alpha;\Phi)$. By 12°, $\mathrm{pr}(\alpha;\Phi) = \phi_3$. By the construction of ξ_2, $\mathrm{t}(\xi_2)$ is a vertex of β. We have $\mathrm{o}(\eta) = \mathrm{t}(\xi_1)$. Therefore, η connects a vertex of τ to a vertex of μ. By construction, η is a simple path. Clearly, η is in $\mathrm{S}(\mu;\Phi)$.

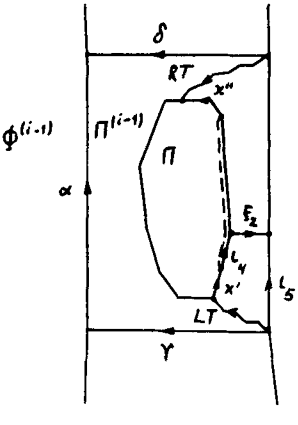


Fig. 117.

Using (7), (10), 12° and 14°, we see that the path τ_1 defined by

$$\tau_1 := \tau'\phi_1\phi_2\iota_2^{-1} \tag{24}$$

is a head of τ such that $t(\tau_1) = o(\iota_2) = t(\xi_1) = o(\eta)$. By 7°, 9°, 14°(β) and the assumption of Case 3, we have

$$\tau_1 = \tau'\phi_1\phi_2\iota_2^{-1} \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 4\frac{1}{2}\cdot 13^{i-j}e_j + 3e_i\right) \subseteq \mathcal{H}\left(\Phi; \sum_{j=1}^{i} \frac{1}{2}\cdot 13^{i+1-j}e_j\right).$$

Since $\mathcal{M}$ satisfies (S_0), τ_1 cannot contain a boundary cycle of Φ and therefore τ_1 is the minimal head of τ such that $t(\tau_1) = o(\eta)$. We have verified (a).

Let ι_5 be the head of β such that $t(\iota_5) = t(\xi_2) = t(\eta)$ (see Fig. 117). In view of (5) and (9), the path μ_0 defined by

$$\mu_0 := \mu'\theta'\theta_1\theta_2\iota_5 \tag{25}$$

is a head of μ such that $t(\mu_0) = t\iota_5) = t(\eta)$ (see Fig. 109).

Using 5°(β), Lemma 28(d), 14°(γ), (24), (25) and the fact that $LT(o(\beta);\Phi) = LT(o(\mu_{j_3});\Phi) = \gamma\, LT(o(\alpha);\Phi)$ and reasoning exactly as in the proof of Theorem 4, we obtain

$$\omega'\tau_1 \underset{i}{\sim} LT(o(\mu);\Phi)^{-1}\mu_0\eta^{-1}.$$

So we have verified (b) too. This completes the proof of (C) in Case. 3. Since Cases 1, 2, 3 exhaust all possibilities, (C) is proved in its entirety.

In similar fashion, one can prove:

(C′) Either $\tau \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j}e_j)$ or there exists a simple path $\eta' \in Br(i)$ in $S(\mu;\Phi)$ connecting a vertex of τ to a vertex of μ and having properties (a′), (b′).

We now deduce assertion (c) of Theorem 7 from (C) and (C′). As in the proof of Theorem 4, assuming that $\tau \notin \mathcal{H}(\Phi; \Sigma_{j=1}^{i} 13^{i+1-j}e_j)$, we see that, for some subpath θ of τ, $\tau = \tau_1\theta\tau_2$. Then, by (2),

$$pr(\mu;\Phi) = \omega'\tau_1\theta\tau_2\omega''. \tag{26}$$

By Lemma 15(f) and Lemma 26(a),

$$pr(\mu;\Phi) \underset{i}{\sim} LT(o(\mu);\Phi)^{-1}\mu\, RT(t(\mu);\Phi). \tag{27}$$

Using (b), (b′), (26) and (27), we obtain

$$\theta \underset{i}{\sim} \eta\xi\eta'^{-1},$$

where ξ is the path obtained by reducing the path $\mu_0^{-1}\mu\mu_0'^{-1}$. Since μ_0 is a head of μ and μ_0' is a tail of μ, ξ is a subpath of μ or μ^{-1}.

This completes the proof of Theorem 7.

§8. Elimination of condition (SC_{n-1})

THEOREM 8. *Let $\mathcal{M} = (M, \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}, <)$ be an ordered n-ranked map satisfying condition* (S_0).

If M is simply-connected, then $\mathcal{M}$ satisfies condition (SC_{n-1}) *(hence also condition* (SC_i) *for any i, $0 \leqq i < n$).*

PROOF. We proceed by induction on the number of regions of M. Assume, then, that the statement is true for any map with less regions.

We shall prove by induction on i that $\mathcal{M}$ satisfies (SC_i), $0 \leqq i < n$. First, we show that $\mathcal{M}$ satisfies (SC_0).

Let Φ be a region in M. If $\mathrm{clos}(\Phi)$ is not simply-connected, there is a closed boundary path ω of Φ such that

(α) ω does not contain a boundary cycle of Φ;

(β) ω is a boundary cycle of some regular simply-connected submap N of M such that $\mathrm{int}(N)$ is connected (see Fig. 118).

Let $\mathcal{U}_i := \mathcal{T}_i \cap \mathrm{Reg}(N)$ and let m be the maximal integer such that $\mathcal{U}_m \neq \emptyset$. Then $\mathcal{N} = (N, \{\mathcal{U}_1, \cdots, \mathcal{U}_m\}, <)$ is an ordered m-ranked map. Since $\mathcal{M}$ satisfies condition (S_0), the same is true of $\mathcal{N}$. Since $\Phi \notin \mathrm{Reg}(N)$, N has less regions than M. Then, by the induction hypothesis, $\mathcal{N}$ satisfies (SC_{m-1}). Hence, there is defined the sequence

$$\mathcal{N}^{(0)} = \mathcal{N}, \mathcal{N}^{(1)}, \cdots, \mathcal{N}^{(m-1)}.$$

Consider $\mathcal{N}^{(m-1)}$. By Corollary 2 to Theorem 4, $\tilde{\mathcal{N}}^{(m-1)}$ satisfies D(8). But $\tilde{\mathcal{N}}^{(m-1)} = (N^{(m-1)}, \{\mathcal{U}_m^{(m-1)}\}, <)$ has only regions of rank 1 (recall that for $\Gamma^{(i)} \in \mathcal{U}_r^{(i)}$

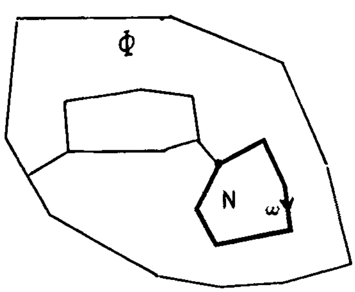


Fig. 118.

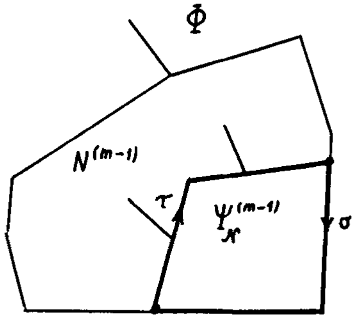


Fig. 119.

one has $\mathrm{rank}(\Gamma^{(i)}) = r - i$ in $\mathcal{N}^{(i)}$). Therefore each inner region of $N^{(m-1)}$ has at least 9 neighbouring regions (see Definition 13).

According to theorem V.4.3 of [1, p. 248], there exist a region $\Psi_{\mathcal{N}}^{(m-1)}$ in $N^{(m-1)}$ and a boundary cycle $\sigma\tau$ of $\Psi_{\mathcal{N}}^{(m-1)}$ such that σ is a subpath of ω and $\tau \in \Psi_{\mathcal{N}}^{(m-1)}(3e_1)$ in $\mathcal{N}^{(m-1)}$ (see Definition 30). (See Fig. 119.) By Corollary 1 to Theorem 4 and 4° of Theorem 4, $\mathrm{pr}_{\mathcal{N}}(\tau;\Psi) \in \mathcal{H}(\Psi; \Sigma_{j=1}^{m} 3\cdot 13^{m-j} e_j)$. The path σ is on the common boundary of $\Psi_{\mathcal{N}}^{(m-1)}$ and Φ.

If $\mathrm{rank}(\Phi) = r \geqq m$, then applying Theorem 5 with $i, k, \Phi, \Psi, r, s, \mu, \tau$ replaced by $m-1, 0, \Psi, \Phi, m, r, \sigma^{-1}$, $\mathrm{pr}_{\mathcal{N}}(\sigma^{-1};\Psi)$, we obtain

$$\mathrm{pr}_{\mathcal{N}}(\sigma;\Psi) \in \mathcal{H}\left(\Psi; \sum_{j=1}^{m-1} 13^{m-j} e_j + e_r\right).$$

If $\mathrm{rank}(\Phi) = r < m$ then, by Theorem 6, $\mathrm{pr}(\sigma;\Psi) \in \mathcal{H}(\Psi; \Sigma_{j=1}^{m-1} 13^{m-j} e_j)$. Using Lemma 7(f) and Lemma 26(b), we obtain that in both cases there is a boundary cycle ψ of Ψ such that

$$\psi \in \mathcal{H}\left(\Psi; \sum_{j=1}^{m-1} 4\cdot 13^{m-j} e_j + 3e_m + e_r\right).$$

This is impossible in view of (S_0). Therefore $\mathrm{clos}(\Phi)$ is simply-connected for any region Φ of M, and so $\mathcal{M}$ satisfies (SC_0).

Now let $n > i > 1$, and assume that $\mathcal{M}$ satisfies (SC_{i-1}). We show that (SC_i) is also satisfied.

Indeed, if this is not the case then, by 5.1, the ordered 2-ranked map

$$\tilde{\mathcal{M}}^{(i-1)} = (M^{(i-1)}, \{\mathcal{T}_i^{(i-1)}, \mathcal{T}_{i+1}^{(i-1)} \cup \cdots \cup \mathcal{T}_n^{(i-1)}\}, <)$$

does not satisfy condition (SC). Then, by 3.5, there exist a region Φ in M, of rank $r > i$, an integer $h \geqq 0$ and a regular submap L of $M^{(i-1)}$, such that

1°. $C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)}) \subseteq L \subseteq C^{h+1}_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$.

2°. L is not simply-connected.

Without loss of generality, we choose h as small as possible and, for this h, choose L with the smallest possible number of regions such that 1° and 2° remain true. By Lemma 11,

3°. $\mathrm{int}(L)$ is connected.

4°. L is distinct from $C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$.

Indeed, if $h = 0$, then $\mathrm{supp}(C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})) = \mathrm{clos}(\Phi^{(i-1)})$. Since $\mathcal{M}$ satisfies (SC_{i-1}), $\mathrm{clos}(\Phi^{(i-1)})$ is simply-connected; therefore, in view of 2°, $\mathrm{clos}(\Phi^{(i-1)}) \neq \mathrm{supp}(L)$.

We know that L and $C^0_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$ are submaps of $M^{(i-1)}$; hence $L \neq C^0_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$.

Let $h > 0$. Then $L = C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$ implies

$$C^{h-1}_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)}) \subseteq L \subseteq C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$$

contradicting the minimality of h. Necessarily, therefore, in this case also $L \neq C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$, as required.

Comparing 1° and 4°, we obtain

5°. There is a region Ψ in M, of rank i, such that $\Psi^{(i-1)} \in \mathcal{L}^{h+1}_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)}) \cap \mathrm{Reg}(L)$.

Let L_1 be the regular submap of L containing all the regions of L except $\Psi^{(i-1)}$. In view of 1° and 5°,

6°. $C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)}) \subseteq L_1 \subseteq C^{h+1}_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$.

The map L_1 contains less regions than L. Therefore, thanks to the minimality property of the number of regions of L, we have:

7°. L_1 is simply-connected.

By Lemma 11, we obtain:

8°. $\mathrm{int}(L_1)$ is connected.

Since $\mathcal{M}$ satisfies (SC_{i-1}), we have:

9°. $\mathrm{clos}(\Psi^{(i-1)})$ is simply-connected.

Since $\Psi^{(i-1)}$ is a region in $M^{(i-1)}$, we have:

10°. $\Psi^{(i-1)}$ is connected.

Furthermore, $\Psi^{(i-1)} \in \mathcal{L}^{h+1}_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)})$, while $C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)}) \subseteq L_1$; therefore by the Corollary to Lemma 10:

11°. $\Psi^{(i-1)}$ and L_1 have at least one common boundary edge.

It is also true that:

12°. $\mathrm{int}(L_1) \cap \Psi^{(i-1)} = \varnothing$.

In view of 7°, 8°, 9°, 10°, 11°, 12°, there exist paths $\omega_1, \omega_2, \omega_3, \omega_4$ such that

13°. $\omega_1\omega_2$ is a p.o.b.c. of L_1.

14°. $\omega_3\omega_4$ is a p.o.b.c. of $\Psi^{(i-1)}$.

15°. $\omega_1\omega_3$ is a p.o.b.c. of the simply-connected submap L_0 of $M^{(i-1)}$ obtained from L by filling in all its holes (i.e. bounded connected components of the complement to $\mathrm{supp}(L)$) (see Fig. 120).

We distinguish between two cases:

(1) $\mathrm{Reg}(L_0) \subseteq \{\Phi^{(i-1)}\} \cup \mathcal{T}_i^{(i-1)}$.

(2) $\mathrm{Reg}(L_0) \not\subseteq \{\Phi^{(i-1)}\} \cup \mathcal{T}_i^{(i-1)}$.

Let us consider each case separately.

Case 1. $\mathrm{Reg}(L_0) \subseteq \{\Phi^{(i-1)}\} \cup \mathcal{T}_i^{(i-1)}$.

Let P be a submap of $M^{(i-1)}$ that fills in some of the holes of L (see Fig. 120).

In other words, int(P) is a bounded connected component of compl(L).

By the construction of P:

16°. P is a regular simply-connected map and int(P) is connected.

Next, there are two paths ν_1, ν_2 such that

17°. ν_1 is a subpath of ω_2, ν_2 is a subpath of ω_1 and $\nu_1\nu_2$ is a boundary cycle of P (see Fig. 121).

Since $\Phi^{(i-1)}$ is a region of L_1, it is not a region of P; hence, by the assumption of Case 1, Reg(P) $\subseteq \mathcal{T}_i^{(i-1)}$.

By Corollary 2 to Theorem 4, $\tilde{\mathcal{M}}^{(i-1)}$ satisfies condition D(8). This means that each inner region $\Gamma^{(i-1)} \in \mathcal{T}_i^{(i-1)}$ of $M^{(i-1)}$ all of whose neighbouring regions belong to $\mathcal{T}_i^{(i-1)}$, has at least 9 neighbouring regions. In particular, each inner region of P has at least 9 neighbouring regions. Then, by theorem V.4.3 of [1, p. 248], we obtain:

18°. There exist a region $\Gamma^{(i-1)}$ in P and a boundary cycle $\sigma\tau$ of $\Gamma^{(i-1)}$ such that σ is a subpath of $\nu_1\nu_2$ and $\tau \in \Gamma^{(i-1)}(3e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$ (see Fig. 122).

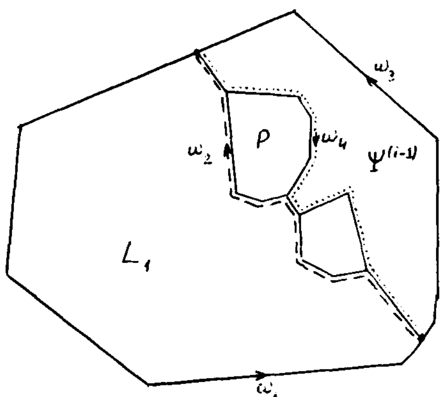


Fig. 120.

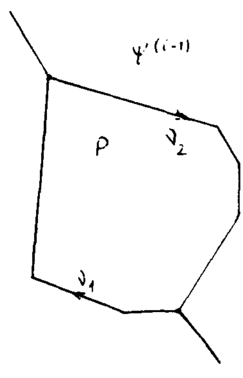


Fig. 121.

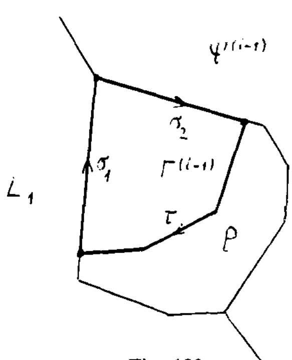


Fig. 122.

We can write $\sigma = \sigma_1\sigma_2$, where σ_1 (σ_2), if non-trivial, is a subpath of ν_1 (of ν_2). If σ_2 is non-trivial, it is on the common boundary of $\Gamma^{(i-1)} \in \mathcal{T}_i^{(i-1)}$ and $\Psi^{(i-1)} \in \mathcal{T}_i^{(i-1)}$. Hence

(1) $$\sigma_2 \in \Gamma^{(i-1)}(e_1) \quad \text{in } \tilde{\mathcal{M}}^{(i-1)}.$$

Now consider σ_1. Let

(2) $$\sigma_1 = \lambda_1\lambda_2 \cdots \lambda_p$$

be the l.h.s factorization of σ_1 in $M^{(i-1)}$ and let

(3) $$\Lambda_1, \Lambda_2, \cdots, \Lambda_p$$

be the corresponding sequence of regions. Denote

(4) $$l_j := \mathrm{d}_{M^{(i-1)}}(\Lambda_j, \Phi^{(i-1)}), \qquad 1 \leqq j \leqq p.$$

Since L_1 is to the left of σ_1 and, by 6°, $L_1 \subseteq C^{h+1}_{\tilde{\mathcal{M}}^{(i-1)}}(\Phi^{(i-1)})$, it follows that

(5) $$l_j \leqq h+1, \qquad 1 \leqq j \leqq p.$$

By Lemma 8(a), $\Gamma^{(i-1)} \in \mathcal{L}_{\tilde{\mathcal{M}}^{(i-1)}}(\Pi^{(i-1)})$ for some region $\Pi^{(i-1)}$. By Lemma 11, $C_{\tilde{\mathcal{M}}^{(i-1)}}(\Pi^{(i-1)})$ is connected. Therefore, necessarily $\Pi^{(i-1)} = \Phi^{(i-1)}$. Thus, $\Gamma^{(i-1)} \in \mathcal{L}_{\tilde{\mathcal{M}}^{(i-1)}}(\Phi^{(i-1)})$. But $\Gamma^{(i-1)}$ is not a region in L_1 and so, in view of 6°, $\mathrm{d}_{M^{(i-1)}}(\Gamma^{(i-1)}, \Phi^{(i-1)}) > h$. On the other hand, for any j,

$$\mathrm{d}_{M^{(i-1)}}(\Gamma^{(i-1)}, \Phi^{(i-1)}) \leqq \mathrm{d}_{M^{(i-1)}}(\Lambda_j; \Phi^{(i-1)}) + 1 = l_j + 1.$$

Comparing these two inequalities, we obtain

(6) $$l_j \geqq h, \qquad 1 \leqq j \leqq p.$$

19°. There is no j, $1 < j < p$, such that $l_{j-1} \leqq l_j$ and $l_{j+1} \leqq l_j$.

Indeed, suppose that there exists j, $1 < j < p$, such that $l_{j-1} \leqq l_j$ and $l_{j+1} \leqq l_j$. Then, as in Lemma 17(d), $\lambda_j = \beta(\Lambda_j)$ and therefore $\beta(\Lambda_j) \in \Lambda_j(e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$, since $\lambda_j = \beta(\Lambda_j)$ is on the common boundary of Λ_j and $\Gamma^{(i-1)} \in \mathcal{T}_i^{(i-1)}$.

By Lemma 22(a), $\gamma(\Lambda_j) \in \Lambda_j(e_1)$ and $\delta(\Lambda_j) \in \Lambda_j(e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$. If $h = 0$ then $\alpha(\Lambda_j) \in \Lambda_j(e_2)$ in $\tilde{\mathcal{M}}^{(i-1)}$; and if $h > 0$ then, by Lemma 22(c), $\alpha(\Lambda_j) \in \Lambda_j(2e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$.

Let $\pi := \alpha(\Lambda_j)^{-1}\gamma(\Lambda_j)^{-1}\beta(\Lambda_j)\delta(\Lambda_j)$. By Lemma 6, π is a boundary cycle of Λ_j. We obtain that if $h = 0$, then $\pi \in \Lambda_j(3e_1 + e_2)$ in $\tilde{\mathcal{M}}^{(i-1)}$, contradicting D(6; 1); and if $h > 0$, then $\pi \in \Lambda_j(5e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$, contradicting D(8). This contradiction shows that there is no j, $1 < j < p$, such that $l_{j-1} \leqq l_j$ and $l_{j+1} \leqq l_j$, as required.

By (5) and (6), $l_j = h$ or $h+1$ for $j = 1, 2, \cdots, p$. Therefore, as we have mentioned in the proof of Lemma 25, if $p>4$ then there is always a j, $1<j<p$, such that $l_{j-1}\leqq l_j$ and $l_{j+1}\leqq l_j$. Hence, in view of 19°, $p\leqq 4$.

If $h>0$, then $\Lambda_j\in\mathcal{T}_i^{(i-1)}$ for any j, and then $\sigma_1\in\Gamma^{(i-1)}(pe_1)\subseteq\Gamma^{(i-1)}(4e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$. Then, by (1) and 18°, the boundary cycle $\sigma_1\sigma_2\tau$ of $\Gamma^{(i-1)}$ belongs to $\Gamma^{(i-1)}(8e_1)$ in $\tilde{\mathcal{M}}^{(i-1)}$, in contradiction to D(8).

If $h=0$, then necessarily $l_j=0$ or 1 for $j=1,2,\cdots,p$, and l_j and l_{j-1} cannot both vanish. Hence, in view of 19°, the only possible sequences $(l_1,\cdots,l_p)$ are the following:

$$(0),(1),(0,1),(1,0),(1,1),(1,0,1).$$

In each of these cases, $\sigma_1\in\Gamma^{(i-1)}(2e_1+e_2)$ in $\tilde{\mathcal{M}}^{(i-1)}$, and then $\sigma_1\sigma_2\tau\in\Gamma^{(i-1)}(6e_1+e_2)$ in $\tilde{\mathcal{M}}^{(i-1)}$; this contradicts D(6; 1). We have thus shown that Case 1 is impossible.

Case 2. $\mathrm{Reg}(L_0)\not\subseteq\{\Phi^{(i-1)}\}\cup\mathcal{T}_i^{(i-1)}$.

Let L_2 be the regular submap of L_0 containing all the regions of L_0 except $\Psi^{(i-1)}$. It follows from 13°, 14° and 15° that

20°. $\omega_1\omega_4^{-1}$ is a p.o.b.c. of L_2 (see Fig. 120).

In view of 7°, 8°, 9°, 10°, ω_1 and ω_4 are simple paths which have no common vertices except for their ends: $\mathrm{o}(\omega_1)=\mathrm{o}(\omega_4)\neq\mathrm{t}(\omega_1)=\mathrm{t}(\omega_4)$. Therefore:

21°. L_2 is simply-connected and $\mathrm{int}(L_2)$ is connected.

Let N denote the submap of M such that $\mathrm{supp}(N)=\mathrm{supp}(L_2)$. Since L_2 is a regular submap of $M^{(i-1)}$, we have:

22°. N is a regular simply-connected $(i-1)$-submap of M such that $\mathrm{int}(N)$ is connected.

Denote $\mathcal{V}_j:=\mathcal{T}_j\cap\mathrm{Reg}(N)$, and let q be the maximal integer such that $\mathcal{V}_q\neq\emptyset$. Then $\mathcal{N}=(N,\{\mathcal{V}_1,\cdots,\mathcal{V}_q\},<)$ is an ordered q-ranked map satisfying (S_0), since $\mathcal{M}$ satisfies (S_0). Since N contains the region Φ of rank $r>i$, we have $q\geqq r>i$. Then map N has less regions than M, since $\Psi\notin\mathrm{Reg}(N)$ and, by 22°, N is simply-connected. Therefore, by the induction hypothesis, $\mathcal{N}$ satisfies (SC_{q-1}). By Lemma 27, $\Gamma_{\mathcal{N}}^{(i-1)}=\Gamma_{\mathcal{M}}^{(i-1)}$ for any region Γ in N of rank $\geqq i$, and therefore $N^{(i-1)}$ is a submap of $M^{(i-1)}$. Since $\mathrm{supp}(N^{(i-1)})=\mathrm{supp}(N)=\mathrm{supp}(L_2)$ and L_2 is a submap of $M^{(i-1)}$, we obtain:

23°. $N^{(i-1)}=L_2$.

We now claim that

24°. $L_1\subseteq C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})$.

Indeed, by 13°, 14°, 15° and 20°, $L\subseteq L_0$. The map L_1 (L_2) is obtained from L (from L_0) by deleting $\Psi^{(i-1)}$ and some of its boundary edges and vertices; hence $L_1\subseteq L_2$.

By 6°,

$$C^h_{\mathcal{M}^{(i-1)}}(\Phi^{(i-1)}) \subseteq L_1 \subseteq L_2 = N^{(i-1)}.$$

Therefore, by Lemma 19,

$$C^{h+1}_{\mathcal{M}^{(i-1)}|}(\Phi^{(i-1)}) \cap N^{(i-1)} \subseteq C^{h+1}_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)}) \subseteq C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})$$

and then, by 6°,

$$L_1 \subseteq C^{h+1}_{\mathcal{M}^{(i-1)}|}(\Phi^{(i-1)}) \cap N^{(i-1)} \subseteq C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)}),$$

as required.

25°. $C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)}) \neq N^{(i-1)}$.

Indeed, by the assumption of Case 2, $\mathrm{Reg}(L_0) \not\subseteq \{\Phi^{(i-1)}\} \cup \mathcal{T}_i^{(i-1)}$. But $\mathrm{Reg}(L_0) = \mathrm{Reg}(L_2) \cup \{\Psi^{(i-1)}\}$ and $\Psi^{(i-1)} \in \mathcal{T}_i^{(i-1)}$, hence $\mathrm{Reg}(N^{(i-1)}) = \mathrm{Reg}(L_2) \not\subseteq \{\Phi^{(i-1)}\} \cup \mathcal{T}_i^{(i-1)}$ while

$$\mathrm{Reg}(C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})) = \mathcal{L}_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)}) \subseteq \{\Phi^{(i-1)}\} \cup \mathcal{T}_i^{(i-1)}.$$

Since $L_1 \subseteq C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)}) \subseteq L_2 = N^{(i-1)}$, it follows from 13°, 14°, 15° and 20° that there is a path ω_5 such that

26°. $\omega_1\omega_5$ is a p.o.b.c. of $C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})$ (we recall that, by the induction hypothesis, $\mathcal{N}$ satisfies (SC_i) and therefore $C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})$ is simply-connected).

Since $C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)}) \neq N^{(i-1)}$, there is a submap Q of $N^{(i-1)}$ which fills in one of the holes in $\mathrm{supp}(C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)})) \cup \mathrm{clos}(\Psi^{(i-1)})$ (see Fig. 123). Let H be the regular submap of N such that $\mathrm{supp}(H) = \mathrm{supp}(Q)$. By the construction of H, we have:

27°. H is a regular simply-connected map and $\mathrm{int}(H)$ is connected.

Since $\Phi^{(i)}_{\mathcal{N}} = \mathrm{int}(C_{\mathcal{N}^{(i-1)}}(\Phi^{(i-1)}))$ and $\mathrm{int}(H)$ is one of the connected components of $\mathrm{int}(N) \setminus \mathrm{clos}(\Phi^{(i)}_{\mathcal{N}})$, we obtain:

28°. H is an i-submap of $\mathcal{N}$.

On the other hand, since $\mathrm{supp}(H) = \mathrm{supp}(Q)$ and Q is a submap of $N^{(i-1)}$, hence also of $M^{(i-1)}$, we have:

29°. H is an $(i-1)$-submap of $\mathcal{M}$.

Since $\mathcal{N}$ satisfies (SC_{q-1}), we have

30°. $\mathrm{clos}(\Phi^{(i)}_{\mathcal{N}})$ is simply-connected.

Hence there is a boundary cycle $\xi_1\xi_2$ of H such that

31°. ξ_1 is a subpath of ω_5 and ξ_2 is a subpath of ω_4 (see Fig. 124).

Let $\mathcal{W}_j := \mathcal{T}_j \cap \mathrm{Reg}(H)$ and let s be the maximal integer such that $\mathcal{W}_s \neq \emptyset$. Then $\mathcal{H} = (H, \{\mathcal{W}_1, \cdots, \mathcal{W}_s\}, <)$ is an ordered s-ranked map satisfying (S_0). Since $\Psi \notin \mathrm{Reg}(H)$, H has less regions than M and so, by 27° and the induction hypothesis, $\mathcal{H}$ satisfies (SC_{s-1}). By 28°, $s \geqq i$.

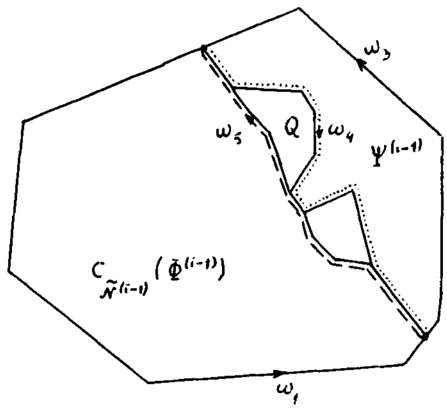


Fig. 123.

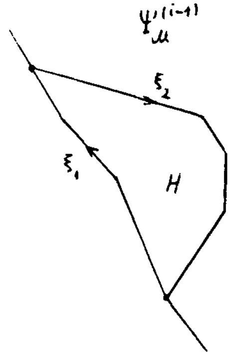


Fig. 124.

Consider the map $\tilde{\mathscr{H}}^{(s-1)}$. By Corollary 2 to Theorem 4, $\tilde{\mathscr{H}}^{(s-1)}$ satisfies D(8). All the regions of $\tilde{\mathscr{H}}^{(s-1)}$ are of rank 1, therefore each inner region of $H^{(s-1)}$ has at least 9 neighbouring regions. By theorem V.4.3 of [1, p. 248], there exist a region $\Pi_{\mathscr{H}}^{(s-1)}$ in $H^{(s-1)}$ and a boundary cycle $\varepsilon\eta$ of $\Pi_{\mathscr{H}}^{(s-1)}$ such that

32°. ε is a subpath of $\xi_1\xi_2$ and $\eta \in \Pi_{\mathscr{H}}^{(s-1)}(3e_1)$ (see Fig. 125).

We can write $\varepsilon = \varepsilon_1\varepsilon_2$ where $\varepsilon_1(\varepsilon_2)$, if non-trivial, is a subpath of ξ_1 (of ξ_2).

By Corollary 1 to Theorem 4, and 4° of Theorem 4,

$$\text{pr}_{\mathscr{H}}(\eta;\Pi) \in \mathscr{H}\left(\Pi; \sum_{j=1}^{s} 3\cdot 13^{s-j} e_j\right). \tag{7}$$

We now apply Theorem 6, with $\mathscr{M}, \mathscr{N}, k, i, \Phi, \Psi, \mu$ replaced by $\mathscr{M}, \mathscr{H}, i-1, s-1, \Pi, \Psi, \varepsilon_2^{-1}$. This gives

$$\text{pr}_{\mathscr{H}}(\varepsilon_2;\Pi) \in \mathscr{H}\left(\Pi; \sum_{j=1}^{s-1} 13^{s-j} e_j\right). \tag{8}$$

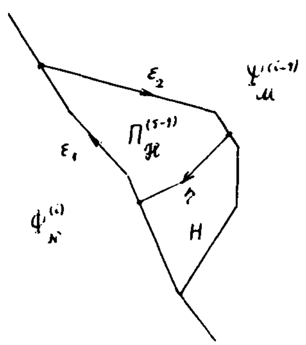


Fig. 125.

If $r = \text{rank}(\Phi) > s-1$, we apply Theorem 5, with $\mathcal{M}, \mathcal{N}, k, i, \Phi, \Psi, \mu$ replaced by $\mathcal{N}, \mathcal{H}, i, s-1, \Pi, \Phi, \varepsilon_1^{-1}$. The result is

$$\text{pr}_{\mathcal{H}}(\varepsilon_1; \Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{s-1} 13^{s-j} e_j + e_r\right). \tag{9}$$

If $r = \text{rank}(\Phi) \leqq s-1$, we apply Theorem 6, with $\mathcal{M}, \mathcal{N}, k, i, \Phi, \Psi, \mu$ replaced by $\mathcal{N}, \mathcal{H}, i, s-1, \Pi, \Phi, \varepsilon_1^{-1}$. Then

$$\text{pr}_{\mathcal{H}}(\varepsilon_1; \Pi) \in \mathcal{H}\left(\Pi; \sum_{j=1}^{s-1} 13^{s-j} e_j\right). \tag{10}$$

Since $\varepsilon_1 \varepsilon_2 \eta$ is a boundary cycle of $\Pi_{\mathcal{H}}^{(s-1)}$ it follows, by Lemma 7(d), (f) and Lemma 26(b), that there is a boundary cycle χ of Π with the property:

if $r = \text{rank}(\Phi) > s-1$ then $\chi \in \mathcal{H}(\Pi; \Sigma_{j=1}^{s-1} 5 \cdot 13^{s-j} e_j + 3e_s + e_r)$, and

if $r = \text{rank}(\Phi) \leqq s-1$ then $\chi \in \mathcal{H}(\Pi; \Sigma_{j=1}^{s-1} 5 \cdot 13^{s-j} e_j + 3e_s)$.

In either case we have a contradiction to (S_0), and so Case 2 is also impossible. This contradiction, in turn, shows that $\mathcal{M}$ satisfies (SC_i). The induction argument is completed and therefore $\mathcal{M}$ satisfies (SC_{n-1}).

The theorem is proved.

§9. Proof of Theorem 3

We have a connected simply-connected ranked map (M, rank) satisfying condition (S_0) and having a reduced boundary cycle α. By the remark in the end of §2, we may assume without loss of generality that M is regular and $\text{int}(M)$ is connected.

Let $\mathcal{T}_i$ be the set of regions of M of rank i. Let n be the maximal integer such that $\mathcal{T}_n \neq \emptyset$. We have $\text{Reg}(M) = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_n$, where $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset$ for $i \neq j$. We introduce a linear order "$<$" on the set $\mathcal{T}_2 \cup \cdots \cup \mathcal{T}_n$, subject to the condition that if $\text{rank}(\Phi) < \text{rank}(\Psi)$ for two regions Φ, Ψ, then also $\Phi < \Psi$. By Definition 12, we obtain an ordered n-ranked map

$$\mathcal{M} = (M, \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n\}, <).$$

(i) By Theorem 8, $\mathcal{M}$ satisfies (SC_{n-1}). Consider the map

$$\tilde{\mathcal{M}}^{(n-1)} = (M^{(n-1)}, \{\mathcal{T}_n^{(n-1)}\}, <).$$

By Corollary 2 to Theorem 4, $\tilde{\mathcal{M}}^{(n-1)}$ satisfies condition D(8). Since all the regions of $M^{(n-1)}$ are of rank 1, this means that each inner region of $M^{(n-1)}$ has at least 9 neighbouring regions in $M^{(n-1)}$. Then, applying theorem V.4.3 of [1, p. 248], we conclude that there exist a region $\Phi^{(n-1)}$ in $M^{(n-1)}$ and a boundary cycle $\phi_1\phi_2$ of $\Phi^{(n-1)}$ such that

1°. ϕ_1 is a subpath of α.

2°. $\phi_2 \in \Phi^{(n-1)}(3e_1)$ in $\bar{\mathcal{M}}^{(n-1)}$ (see Fig. 126).

If $\phi_1 \notin \bigcap_{h=1}^{n-1} \mathcal{G}^h(\Sigma_{j=1}^{h-1} 5\cdot 13^{h-j}e_j + 4e_h)$ then, for some i, $1 \leqq i \leqq n-1$, $\phi_1 \notin \mathcal{G}^i(\Sigma_{j=1}^{i-1} 5\cdot 13^{i-j}e_j + 4e_i)$. By Definition 34, this means that there exist

(α) a factorization $\phi_1 = \phi_1'\beta\phi_1''$;

(β) simple paths $\sigma, \tau \in \mathrm{Br}(i-1)$;

(γ) a boundary path γ of some region Ψ in M, of rank i, such that

(δ) $\beta \sim_{i-1} \sigma^{-1}\gamma\tau$

and

(ε) either γ contains a boundary cycle of Ψ or, for some δ, $\gamma\delta$ is a boundary cycle of Ψ and

$$\delta \in \mathcal{H}\left(\Psi; \sum_{j=1}^{i-1} 5\cdot 13^{i-j}e_j + 4e_i\right).$$

Since ϕ_1 is a subpath of α, β is also a subpath of α. In this case part (i) of Theorem 3 is proved.

Assume now that $\phi_1 \in \bigcap_{h=1}^{n-1} \mathcal{G}^h(\Sigma_{j=1}^{h-1} 5\cdot 13^{h-j}e_j + 4e_h)$. By Lemma 7(d), (f) and Lemma 26(b), there exists a boundary cycle $\sigma_1\sigma_2$ of Φ such that

3°. $\sigma_1(\sigma_2)$, if non-trivial, is a subpath of $\mathrm{pr}(\phi_1, \Phi)$ (of $\mathrm{pr}(\phi_2; \Phi)$).

4°. If ϕ_2 is trivial then σ_2 is trivial.

In view of 2°, it follows from Corollary 1 to Theorem 4 and 4° of Theo' m 4 that

$$\sigma_2 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{n} 3\cdot 13^{n-j}e_j\right). \tag{1}$$

Then, because of (S_0), $\sigma_1 \notin \mathcal{H}(\Phi; \Sigma_{j=1}^{n-1} 13^{n-j}e_j)$. Therefore, applying Theorem 7 with i, μ, τ replaced by $n-1, \phi_1, \sigma_1$, we conclude that there are two simple paths

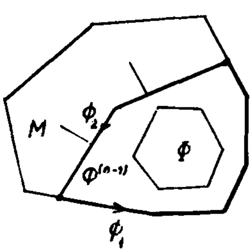


Fig. 126.

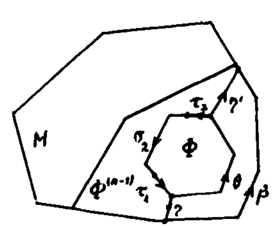


Fig. 127.

$\eta, \eta' \in \mathrm{Br}(n-1)$, each connecting a vertex on σ_1 to a vertex on ϕ_1, with the following properties:

5°. Let τ_1 be the head of σ_1 such that $\mathrm{t}(\tau_1) = \mathrm{o}(\eta)$ and τ_2 the tail of σ_1 such that $\mathrm{o}(\tau_2) = \mathrm{o}(\eta')$. Then $\tau_1, \tau_2 \in \mathscr{H}(\Phi; \Sigma_{j=1}^{n-1} \frac{1}{2} \cdot 13^{n-j} e_j)$.

6°. $\sigma_1 = \tau_1 \theta \tau_2$ for some subpath θ of σ_1. Furthermore, for some subpath β of ϕ_1 or ϕ_1^{-1} connecting $\mathrm{t}(\eta)$ to $\mathrm{t}(\eta')$, $\theta \sim_i \eta\beta\eta'^{-1}$ (see Fig. 127). By (1) and 5°,

$$\tau_2\sigma_2\tau_1 \in \mathscr{H}\left(\Phi; \sum_{j=1}^{n-1} 4 \cdot 13^{n-j} e_j + 3e_n\right) \subseteq \mathscr{H}\left(\Phi; \sum_{j=1}^{n-1} 5 \cdot 13^{n-j} e_j + 4e_n\right).$$

Taking $i := n-1$, $\gamma := \theta$, $\delta := \tau_2\sigma_2\tau_1$, $\sigma := \eta$, $\tau := \eta'$ we see that part (i) of Theorem 3 is proved.

(ii) Let s, $0 \leqq s < n$, be the minimal integer for which there exist a region $\Psi^{(s)}$ in $M^{(s)}$ and a p.o. boundary cycle $\psi_1\psi_2$ of $\Psi^{(s)}$ such that

7°. ψ_1 is a subpath of α.

8°. Either $\psi_2 \in \Psi^{(s)}(4e_1)$ or $\psi_2 \in \Psi^{(s)}(2e_1 + e_2)$ in $\tilde{\mathscr{M}}^{(s)}$.

The existence of this s follows from the fact, verified in the proof of part (i) of Theorem 3, that there exist a region $\Phi^{(n-1)}$ in $M^{(n-1)}$ and a boundary cycle $\phi_1\phi_2$ of $\Phi^{(n-1)}$ satisfying conditions 1° and 2°.

If $\psi_2 \in \Psi^{(s)}(2e_1 + e_2)$ in $\tilde{\mathscr{M}}^{(s)}$ then, by 5.1, $\psi_2 \in \Psi^{(s)}(2e_1 + e_t)$ in $\mathscr{M}^{(s)}$ for some $t > 1$. By Corollary 1 to Theorem 4 and 4° of Theorem 4, we obtain

9°. If $\psi_2 \in \Psi^{(s)}(4e_1)$ in $\tilde{\mathscr{M}}^{(s)}$, hence in $\mathscr{M}^{(s)}$, then

$$\mathrm{pr}(\psi_2; \Psi) \in \mathscr{H}\left(\Psi; \sum_{j=1}^{s+1} 4 \cdot 13^{s+1-j} e_j\right);$$

if $\psi_2 \in \Psi^{(s)}(2e_1 + e_t)$ in $\mathscr{M}^{(s)}$, then

$$\mathrm{pr}(\psi_2; \Psi) \in \mathscr{H}\left(\Psi; \sum_{j=1}^{s} 3 \cdot 13^{s+1-j} e_j + 2e_{s+1} + e_{s+t}\right).$$

Let π_1 be the maximal head of ψ_1 such that π_1 is a head of $\mathrm{RT}(\mathrm{o}(\psi_1); \Psi)$. Then $\psi_1 = \pi_1\psi_0$ for some ψ_0 and let π_2 be the maximal tail of ψ_0 such that π_2^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\psi_1); \Psi)$. Then, for some boundary path ψ of $\Psi^{(s)}$,

$$\psi_1 = \pi_1\psi\pi_2 \tag{2}$$

(see Fig. 128).

Since $\mathrm{o}(\psi_1) = \mathrm{t}(\psi_2)$ and $\mathrm{t}(\psi_1) = \mathrm{o}(\psi_2)$, by Definitions 19, 27 and 32, $\mathrm{pr}(\psi_2; \Psi) = \mathrm{pr}(\pi_2\psi_2\pi_1; \Psi)$. By 9° and ($\mathrm{S}_0$), $\mathrm{pr}(\psi_2; \Psi)$ cannot contain a boundary cycle of Ψ; therefore, in view of Lemma 7(f) and Lemma 26(b):

10°. ψ is non-trivial.

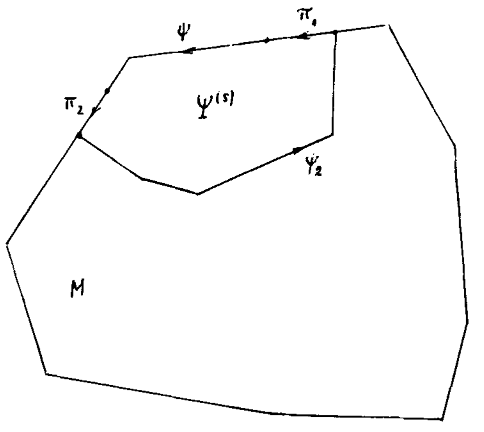


Fig. 128.

We now show by induction on i that ψ is on the boundary of $\Psi^{(s-i)}$, $i=0,1,\cdots,s$. If $i=0$, there is nothing to prove. Let $i>0$, $i<s$. By the induction hypothesis, ψ is on the boundary of $\Psi^{(s-i+1)}$.

11°. Let $\Pi^{(s-i)}$ be a region in $C_{\bar{\mathcal{M}}^{(s-i)}}(\Psi^{(s-i)})$ and σ a subpath of ψ which is a boundary path of $\Pi^{(s-i)}$. If $\Pi^{(s-i)}\neq\Psi^{(s-i)}$, then $\sigma\neq\beta(\Pi^{(s-i)})$.

Indeed, denote $\alpha_1:=\alpha(\Pi^{(s-i)})$, $\beta_1:=\beta(\Pi^{(s-i)})$, $\gamma_1:=\gamma(\Pi^{(s-i)})$, $\delta_1:=\delta(\Pi^{(s-i)})$. Then, by Lemma 6 and Definition 26, $\delta_1\alpha_1^{-1}\gamma_1^{-1}\beta_1$ is a boundary cycle of $\Pi^{(s-i)}$ (see Fig. 129). If $d_{M^{(s-i)}}(\Pi^{(s-i)},\Psi^{(s-i)})=1$ then, by Lemma 22(b), $\delta_1\alpha_1^{-1}\gamma_1^{-1}\in\Pi^{(s-i)}(2e_1+e_2)$ in $\bar{\mathcal{M}}^{(s-i)}$. If $d_{M^{(s-i)}}(\Pi^{(s-i)},\Psi^{(s-i)})>1$ then, by Corollary 2 to Theorem 4 and Lemma 22(c), $\delta_1\alpha_1^{-1}\gamma_1^{-1}\in\Pi^{(s-i)}(4e_1)$ in $\bar{\mathcal{M}}^{(s-i)}$. Therefore, if $\sigma=\beta_1$ then $\Pi^{(s-i)}$ and $\beta_1\delta_1\alpha_1^{-1}\gamma_1^{-1}$ satisfy conditions 7°, 8° with Ψ, s, ψ_1, ψ_2 replaced by $\Pi, s-i, \beta_1, \delta_1\alpha_1^{-1}\gamma_1^{-1}$, contradicting the minimality of s. Thus, $\sigma\neq\beta(\Pi^{(s-i)})$, as required.

We now apply Proposition 1 with M,Φ,μ replaced by $M^{(s-i)},\Psi^{(s-i)},\psi$. This gives a factorization $\psi=\psi'\psi''\psi'''$ and, if ψ'' is non-trivial, a further factorization $\psi''=\nu_1\nu_2\cdots\nu_h$ such that

12°. ψ' is a head of $\mathrm{RT}(\mathrm{o}(\psi);\Psi)$.

13°. ψ'''^{-1} is a head of $\mathrm{LT}(\mathrm{t}(\psi);\Psi)$.

14°. If ψ'' is non-trivial, then ψ'' is on the boundary of $(\Psi^{(s-i)})^1$; if ν_j is not on the boundary of $\Psi^{(s-i)}$ for some j, then $\nu_j=\beta(\Gamma_j^{(s-i)})$ for some $\Gamma_j^{(s-i)}\in\mathcal{L}^1_{\bar{\mathcal{M}}^{(s-i)}}(\Psi^{(s-i)})$.

By the construction of π_1, $\pi_1\psi'$ is a head of $\mathrm{RT}(\mathrm{o}(\psi_1);\Psi)$. Therefore, by the maximality of π_1, it follows that ψ' is trivial. Similarly, ψ''' is trivial, and hence $\psi=\psi''$.

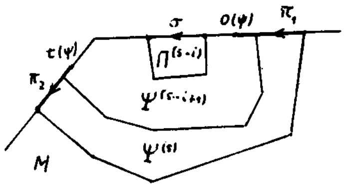


Fig. 129.

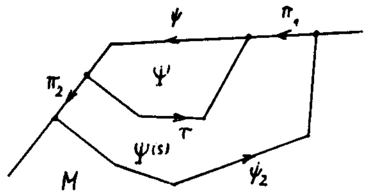


Fig. 130.

Comparing 10°, 11° and 14°, we conclude that ψ is on the boundary of $\psi^{(s-i)}$. This completes the induction argument. Thus, ψ is on the boundary of Ψ.

Let τ be a boundary path of Ψ such that $\psi\tau$ is a boundary cycle of Ψ (see Fig. 130). Since π_1 is a head of $\mathrm{RT}(\mathrm{t}(\psi_2);\Psi)$ and $\mathrm{t}(\pi_1)$ belongs to the boundary of Ψ, we obtain $\pi_1=\mathrm{RT}(\mathrm{t}(\psi_2);\Psi)$. Similarly, $\pi_2^{-1}=\mathrm{LT}(\mathrm{o}(\psi_2);\Psi)$. Then, by Lemma 15(f) and Lemma 26(a), $\tau=\mathrm{pr}(\psi_2;\Psi)$.

We have thus determined a region Ψ and a boundary cycle $\psi\tau$ of Ψ such that ψ is a subpath of α and, by 9°, $\tau=\mathrm{pr}(\psi_2;\Psi)$ belongs either to $\mathcal{H}(\Psi;\Sigma_{j=1}^{s+1}4\cdot 13^{s+1-j}e_j)$ or to $\mathcal{H}(\Psi;\Sigma_{j=1}^{s}3\cdot 13^{s+1-j}e_j+2e_{s+1}+e_{s+t})$. Take $k:=\mathrm{rank}(\Psi)$. Then $s<k$ because $\Psi^{(s)}$ is a region in $M^{(s)}$. Hence either $\tau\in\mathcal{H}(\Psi;\Sigma_{j=1}^{k}4\cdot 13^{k-j}e_j)$ or $\tau\in\mathcal{H}(\Psi;\Sigma_{j=1}^{k-1}3\cdot 13^{k-j}e_j+2e_k+e_{s+t})$ if $s+t>k$, as required. This completes the proof of part (ii).

(iii) We prove by induction on $n-i$ that $\mathrm{card}(\mathcal{T}_i)$ is effectively bounded in terms of the length $|\alpha|$ of the boundary cycle α of M and the maximum l_0 of lengths of boundary cycles of regions of M.

Let $n-i=0$. Consider the map $M^{(n-1)}$. As we have shown, each inner region of $M^{(n-1)}$ has at least 9 neighbouring regions. Therefore, by the "area theorem" ([1], p. 260), $\mathrm{card}(\mathrm{Reg}(M^{(n-1)}))=\mathrm{card}(\mathcal{T}_n^{(n-1)})=\mathrm{card}(\mathcal{T}_n)$ is effectively bounded in terms of $|\alpha|$.

Let $n-i>0$. Consider the ordered 2-ranked map

$$\tilde{\mathcal{M}}^{(i-1)}=(M^{(i-1)},\{\mathcal{T}_i^{(i-1)},\mathcal{T}_{i+1}^{(i-1)}\cup\cdots\cup\mathcal{T}_n^{(i-1)}\},<).$$

By the induction hypothesis, $\mathrm{card}(\mathcal{T}_{i+1}^{(i-1)}\cup\cdots\cup\mathcal{T}_n^{(i-1)})$ is effectively bounded in terms of $|\alpha|$ and l_0. Define

$$\mathcal{U}:=\{\Phi^{(i-1)}\,|\,\Phi^{(i-1)}\in\mathcal{T}_i^{(i-1)},\mathrm{ind}(\Phi^{(i-1)})\leqq 2e_1+2e_2\text{ in }\tilde{\mathcal{M}}^{(i-1)}\}.$$

By Proposition 3, $\mathrm{card}(\mathcal{T}_i^{(i-1)}\setminus\mathcal{U})$ is effectively bounded in terms of $|\alpha|$ and $\mathrm{card}(\mathcal{T}_{i+1}^{(i-1)}\cup\cdots\cup\mathcal{T}_n^{(i-1)})$. Therefore, it is enough to prove the following:

(3) $$\mathrm{card}(\mathcal{U})\leqq 2l_0\mathrm{card}(\mathcal{T}_{i+1}^{(i-1)}\cup\cdots\cup\mathcal{T}_n^{(i-1)}).$$

Consider a region $\Phi^{(i-1)} \in \mathcal{U}$. By Definitions 29 and 31, there exists a positively oriented boundary cycle $\mu\nu$ of $\Phi^{(i-1)}$ such that:

15°. $\nu \in \Phi^{(i-1)}(2e_1 + e_2)$ in $\tilde{\mathcal{M}}^{(i-1)}$.

16°. μ is on the common boundary of $\Phi^{(i-1)}$ and some region $\Psi^{(i-1)} \in \mathcal{T}_{i+1}^{(i-1)} \cup \cdots \cup \mathcal{T}_n^{(i+1)}$.

Furthermore, we have

17°. $\mathrm{pr}(\mu;\Phi) \notin \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$.

Indeed, by Lemma 7(d), (f) and Lemma 26(b), there exists a boundary cycle $\sigma_1\sigma_2$ of Φ such that σ_1 is a subpath of $\mathrm{pr}(\mu;\Phi)$ and σ_2 is a subpath of $\mathrm{pr}(\nu;\Phi)$. By 15° and 5.1, $\nu \in \Phi^{(i-1)}(2e_1 + e_t)$ in $\mathcal{M}^{(i-1)}$ for some $t > 1$; then by Corollary 1 to Theorem 4,

$$\sigma_2 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 3\cdot 13^{i-j}e_j + 2e_i + e_{i+t-1}\right).$$

If $\mathrm{pr}(\mu;\Phi) \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ then also $\sigma_1 \in \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ and then

$$\sigma_1\sigma_2 \in \mathcal{H}\left(\Phi; \sum_{j=1}^{i-1} 4\cdot 13^{i-j}e_j + 2e_i + e_{t+i-1}\right),$$

contradicting (S_0). Thus, $\mathrm{pr}(\mu;\Phi) \notin \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$.

We apply now Theorem 4, with $i, \omega', \tau, \omega''$ replaced by $i-1$, $\mathrm{o}(\mathrm{pr}(\mu;\Phi))$, $\mathrm{pr}(\mu;\Phi)$, $\mathrm{t}(\mathrm{pr}(\mu;\Phi))$. Since $\mathrm{pr}(\mu;\Phi) \notin \mathcal{H}(\Phi; \Sigma_{j=1}^{i-1} 13^{i-j}e_j)$ and $\mathrm{rank}(\Phi) = i < \mathrm{rank}(\Psi)$, we obtain by (A(α)), (A(γ)) that there exists a vertex $\mathrm{t}(\eta)$ of μ which is a common vertex of $\mathrm{bd}(\Psi)$ and $\mathrm{bd}(\Psi^{(i-1)})$.

We assign to any region $\Phi^{(i-1)} \in \mathcal{U}$ a triple $(\Psi^{(i-1)}, \mu, v)$ where

(α) $\Psi^{(i-1)} \in \mathcal{T}_{i+1}^{(i-1)} \cup \cdots \cup \mathcal{T}_n^{(i-1)}$;

(β) μ is a non-trivial path on the common boundary of $\Phi^{(i-1)}$ and $\Psi^{(i-1)}$;

(γ) v is a vertex of μ and $v \in \mathrm{bd}(\Psi) \cap \mathrm{bd}(\Psi^{(i-1)})$.

Let $\Phi_1^{(i-1)}$ and $\Phi_2^{(i-1)}$ be two distinct regions in $\mathcal{U}$ and let $(\Psi_1^{(i-1)}, \mu_1, v_1)$, $(\Psi_2^{(i-1)}, \mu_2, v_2)$ be the corresponding triples. If $\Psi_1^{(i-1)} = \Psi_2^{(i-1)}$, then there are only the following possibilities for v_1 and v_2 to coincide:

$$\mathrm{t}(\mu_1) = v_1 = v_2 = \mathrm{o}(\mu_2), \qquad \mathrm{o}(\mu_1) = v_1 = v_2 = \mathrm{t}(\mu_2)$$

for μ_1 and μ_2 have no (non-oriented) edges in common.

Let $\Psi \in \mathcal{T}_{i+1} \cup \cdots \cup \mathcal{T}_n$ and let ω be a boundary cycle of Ψ. Since the number of distinct vertices v appearing in triples of the type $(\Psi^{(i-1)}, \mu, v)$ with the same $\Psi^{(i-1)}$ cannot exceed $|\omega|$, there are at most $2|\omega|$ such triples. We have $|\omega| \leqq l_0$ and therefore the total number of triples $(\Psi^{(i-1)}, \mu, v)$ cannot exceed $2l_0\mathrm{card}(\mathcal{T}_{i+1} \cup \cdots \cup \mathcal{T}_n)$.

In view of (β), to distinct regions $\Phi_1, \Phi_2 \in \mathcal{U}$ are assigned distinct triples and therefore

$$\mathrm{card}(\mathcal{U}) \leqq 2l_0 \mathrm{card}(\mathcal{T}_{i+1} \cup \cdots \cup \mathcal{T}_n).$$

Since $\mathrm{card}(\mathcal{T}_j) = \mathrm{card}(\mathcal{T}_j^{(i-1)})$ for $j \geqq i$, (3) is proved. This completes the induction. The number of regions of M is thus effectively bounded in terms of l_0 and $|\alpha|$. This proves part (iii).

The theorem is proved.

Acknowledgements

The starting point for this paper was a talk I had with Professor D. Kazhdan; in fact, if not for that talk, this paper would never have appeared. I am greatly indebted to him.

I should like to express my appreciation and most cordial thanks to Professor S. A. Amitsur for his constant attention and encouragement; many improvements throughout the paper are the fruit of my numerous discussions with him.

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